ISRAEL JOURNAL OF MATHEMATICS, Vol. 48, No. 4, 1984

THE NONSTATIONARY IDEAL ON $\aleph_2$

BY

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ABSTRACT

We construct a model in which the filter of ω-closed unbounded subsets of $\aleph_2$ is precipitous and a model in which the filter of closed unbounded subsets of $\aleph_2$ is precipitous. For the first model we need a measurable cardinal, and for the second a measurable cardinal of order 2. Both results are equiconsistent.

Let I be a nontrivial κ-complete ideal over some uncountable cardinal κ. Define $R(I)$ to be the notion of forcing with I-positive subsets of κ as conditions. For X, $Y \in R(I)$, X is stronger than Y iff $(X - Y) \in I$.

Jech and Prikry introduced the notion of precipitous ideal. I is precipitous iff $\kappa \Vdash_{R(I)} V^\kappa/\mathbf{G}$ is well founded, where $\mathbf{G}$ is the canonical name of a generic ultrafilter.

If $\mathcal{F}$ is the dual filter of I let us say that $\mathcal{F}$ is precipitous if I is such and denote by $R(\mathcal{F})$ the forcing notion $R(I)$.

Jech, Magidor, Mitchell and Prikry [7] proved that the following is equiconsistent:

(1) There is a measurable cardinal.

(2) There is a precipitous ideal on $\aleph_1$.

(3) $NS_{\aleph_1}$ (the nonstationary ideal on $\aleph_1$) is precipitous.

The idea for making $NS_{\aleph_1}$ precipitous was to collapse a measurable cardinal to $\aleph_1$ by the Levy collapse and then iterate the forcing for adding closed unbounded subsets of $\aleph_1$. This construction can be extended to obtain a model in which the filter of ω_1-closed unbounded subsets of $\aleph_2$ is precipitous Already for getting a normal precipitous filter D on $\aleph_2$ s.t. $\{\delta < \aleph_2 \mid \operatorname{cf} \delta = \aleph_0\} \in D$ some new approach is needed. S. Shelah, by revised countable support (RCS) iteration of a variant of Namba forcing below a measurable, built such a filter.

We are producing a model in which the filter of ω-closed unbounded subsets of $\aleph_2$ is precipitous and a model in which $NS_{\aleph_2}$ is precipitous. For the first model

Received April 26, 1982 and in revised form April 1, 1984

we need a measurable cardinal, and for the second a measurable cardinal of order 2 (i.e. a normal measure which is concentrated on measurable cardinals). Both results are equiconsistent.

We don't know whether NS_κ can be precipitous for $\kappa > \aleph_2$ or even if the ideal of ω-closed subsets of $\aleph_3$ can be precipitous.

Our work was inspired by Shelah's solution of Friedman's Problem. We are grateful to Saharon Shelah for explaining to us his proof and to Menachem Magidor for the helpful discussions we had on the subject.

Part I. The Filter of ω-Closed Unbounded Subsets of $\aleph_2$

In this part we prove the following:

THEOREM I. *If "ZFC + there is a measurable cardinal" is consistent then so is "ZFC + the filter of ω-closed unbounded subsets of $\aleph_2$ is precipitous".*

We start with a model of ZFC + G.C.H. and a measurable cardinal κ. Let V denote our ground model. Let $\mathcal{U}$ be a normal κ-complete ultrafilter over κ and $j : V \to V^\kappa/\mathcal{U}$ be the elementary embedding defined by $\mathcal{U}$. We shall identify the ultrapower $V^\kappa/\mathcal{U}$ with its transitive collapse N.

1. The diamond over κ

We need the special kind of diamond over κ in V. It is $\diamondsuit_\kappa = \langle S_\alpha \mid \alpha \in B\rangle$, $B \notin \mathcal{U}$, every $\alpha \in B$ is weakly compact and the following holds:

(*) for every $A \subseteq \kappa$ and

Π^1_2-sentence $\varphi(\cdot)$ if

$$\langle V_\kappa, \in, A\rangle \models \varphi(A) \text{ then } \{\alpha \in B \mid A \cap \alpha = S_\alpha \text{ and } \langle V_\alpha, \in, S_\alpha\rangle \models \varphi(S_\alpha)\}$$
is stationary.

Such a kind of $\diamondsuit$ was used by S. Shelah in his paper [12] but for Π^1_1-sentences.

We shall present here a well known construction of such kinds of diamond over a measurable.

First let us define it on all weakly compact cardinals below κ. The definition is by induction. Suppose $\langle S_\nu \mid \nu < \beta\rangle$ is built. Let α be the least weakly compact cardinal $\geqq \beta$. Now, if there is a set $A \subseteq \alpha$ and Π^1_2-sentence $\varphi(\cdot)$ so that $\langle V_\alpha, \in, A\rangle \models \varphi(A)$ and $\{\nu < \alpha \mid A \cap \nu = S_\nu \text{ and } \langle V_\nu, \in, S_\nu\rangle \models \varphi(S_\nu)\}$ is nonstationary, then let S_α be some such A. Otherwise let $S_\alpha = \{-1\}$.

PROPOSITION 1.1. *Such defined* $\langle S_\alpha \mid \alpha < \kappa$ *and* α *is weakly compact*$\rangle$ *satisfies* (*).

PROOF. Suppose not. Then there are $A \subseteq \kappa$ and Π^1_2-sentence $\varphi(\cdot)$ so that $\langle V_\kappa, \in, A\rangle \models \varphi(A)$ but $\{\alpha < \kappa \mid A \cap \alpha = S_\alpha$ and $\langle V_\alpha, \in, S_\alpha\rangle \models \varphi(S_\alpha)\}$ is nonstationary.

Now look in N. Since V_κ, $A \in N$, S_κ cannot be $\{-1\}$. So S_κ is equal to some such A. Then

$$j(A) \cap \kappa = A = S_\kappa \quad \text{and} \quad \langle V_\kappa, \in, S_\kappa\rangle \models \varphi(S_\kappa).$$

Hence

$$\{\alpha < \kappa \mid A \cap \alpha = S_\alpha \text{ and } \langle V_\alpha, \in, S_\alpha\rangle \models \varphi(S_\alpha)\} \in \mathcal{U},$$

and so it is stationary in V. Hence it is stationary also in N, which is impossible. □

Note that it follows from the proof that in N, $S_\kappa = \{-1\}$. So for a set of α's in $\mathcal{U}$, $S_\alpha = \{-1\}$. Let us define B to be the set of all weakly compact $\beta < \kappa$ so that $S_\beta \subseteq \beta$.

2. The preparation forcing

The property of a set A to be positive in a Wc_κ-weakly compact filter over κ (i.e. the filter generated by the sets

$$\{\alpha < \kappa \mid \langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi(R \cap V_\alpha)\}$$

for some $R \subseteq V_\kappa$ and Π^1_1-sentence φ s.t. $\langle V_\kappa, \in, R\rangle \models \varphi(R)$), see [8], can be expressed as

$$\sigma(A) \leftrightarrow [\forall R \subseteq \kappa\, \forall n \in \omega (X_{11}(R, n) \rightarrow \exists \alpha \text{ limit } \alpha \in A$$
$$(\langle V_\alpha, \in, R \cap V_\alpha\rangle \models X_{11}(R \cap \alpha, n)))]$$

where $X_{11}(\cdot,\cdot)$ is the universal Π^1_1-formula, so that for any Π^1_1-formula $\varphi(\cdot)$, there is an integer n so that for any limit α and $R \subseteq V_\alpha$,

$$\langle V_\alpha, \in, R\rangle \models \varphi(R) \qquad \text{iff} \langle V_\alpha, \in, R\rangle \models X_{11}(R, n).$$

See Levy [10] or Devlin [4].

It follows that σ is a Π^1_2-sentence.

So for many α's S_α is positive in Wc_α.

Let us define a revised countable support iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i \mid i < \kappa\rangle$, $|P_i| \leqq \aleph_{i+1}$. We refer to [12], [13] or [14] for the definitions and the motivations. $\underset{\sim}{Q}_i$, for $i < \kappa$, is defined as follows. We consider three cases.

Case 1. i is not a strongly inaccessible cardinal. Set $\underset{\sim}{Q}_i$ to be the Levy collapse of $2^{\aleph_i}$ to $\aleph_1$.

Case 2. (i is a strongly inaccessible and $i \notin B$), or ($i \in B$ and (S_i is not positive in Wc_i, or for some $\alpha \in S_i$, α is not an inaccessible cardinal, or $S_i \cap B = \emptyset$)), where B is the set on which $\diamondsuit$ works.

Let then $\underset{\sim}{Q}_i$ be the variant of Namba forcing for changing the cofinality of both i and i^+ to $\aleph_0$. In our case, after we forced with P_i, i became $\aleph_2$ and $i^+ = \aleph_3$. Let us denote this forcing by $\mathrm{Nm}'_{\aleph_2,\aleph_3}$. It will be the set $\{T \mid T$ is a subtree of ${}^{\omega>}\aleph_3$, so that above each $\eta \in T$ there are $\nu_1, \nu_2 \in T$ so that $|\mathrm{Suc}_T \nu_2| = \aleph_3$, $|\mathrm{Suc}_T \nu_1| = \aleph_2$ and $\mathrm{Suc}_T \nu_1 \subseteq \aleph_2\}$. For $T_1, T_2 \in \mathrm{Nm}'_{\aleph_2,\aleph_3}$ we say T_1 is stronger than T_2 if T_1 is a subtree of T_2.

Case 3. $i \in B$, S_i is positive in Wc_i, $S_i \cap B = \emptyset$ and, for every $\alpha \in S_i$, α is an inaccessible cardinal in V.

Let then $\underset{\sim}{Q}_i = P^*[S_i]$ where $P^*[S_i]$ will be the set of all ω-closed subsets c of S_i so that for every limit point β of c, $c \cap \beta$ intersects with every closed unbounded subset of β, which belongs to $V[\overset{\circ}{P}_\beta]$, where following Shelah, we denote by $\overset{\circ}{P}$ a generic subset of P. The ordering on $P^*[S_i]$ is defined as follows: $c_1 \geqq c_2$ if c_1 is an end extension of c_2.

Let $P_\kappa = R \lim \bar{\underset{\sim}{Q}}$.

Let us show that Nm'_{i,i^+} and $P^*[S_i]$ satisfy some nice properties. Then we shall apply [14] and [5] to obtain that

(a) P_κ does not add new subsets of ω,

(b) for every strongly inaccessible i, P_i satisfies i-c.c.

First let us consider $\mathrm{Nm}'_{\aleph_2,\aleph_3}$.

LEMMA 2.1. *$\mathrm{Nm}'_{\aleph_2,\aleph_3}$ satisfies the S-condition for any S s.t. $\{\aleph_2, \aleph_3\} \subseteq S$.*

PROOF. Let us define the function F. For a point η where we are using F to determine $\mathrm{Suc}(\eta)$, I_η and $f(\eta')$ for any immediate successor η' of η, $f(\eta)$ is already known and it is a condition in $\mathrm{Nm}'_{\aleph_2,\aleph_3}$. Also we know for which $l < n$ and $k <$ height of η, $\eta \restriction k$ belongs to the lth front. If there is the maximal $k <$ height η such that $\eta \restriction k$ belongs to some front, let the index of this front be l_η.

If there is such l_η, and l_η is an even number, or for any $k <$ height of η $\eta \restriction k$ does not belong to a front, then let us find a point ν_η of minimal height in $f(\eta)$ such that $\mathrm{Suc}_{f(\eta)}(\nu_\eta) \subseteq \aleph_2$ and $|\mathrm{Suc}_{f(\eta)}(\nu_\eta)| = \aleph_2$. Let $\mathrm{Suc}_T(\eta)$ be $\{\eta{}^\frown\langle\alpha\rangle \mid \nu_\eta{}^\frown\langle\alpha\rangle \in f(\eta)\}$, and I_η be $\{A \subseteq \mathrm{Suc}_T(\eta) \mid |A| < \aleph_2\}$ and for each $\eta{}^\frown\langle\alpha\rangle$ in $\mathrm{Suc}_T(\eta)$ let $f(\eta{}^\frown\langle\alpha\rangle)$ be the subtree of $f(\eta)$ which is defined by $\nu_{\eta^\frown\langle\alpha\rangle}$. If l_η is odd then take ν_η to be a point of minimal height in $f(\eta)$ s.t.

$|\mathrm{Suc}_{f(\eta)}(\nu_\eta)| = \aleph_3$. Let us define $\mathrm{Suc}_T(\eta)$, $f(\eta{}^\frown\langle\alpha\rangle)$ as above, and let $I\eta$ be $\{A \subseteq \mathrm{Suc}_T(\eta) \mid |A| < \aleph_3\}$.

The proof that such a defined strategy F works is the same as in usual Nm′ forcing; see [13] or [14].

Now suppose that on a step i we force with $P^*[S_i]$. Then $i \in B$, S_i is positive in the weakly compact filter on i, and for every $\alpha \in S_i$ $\emptyset \Vdash_{P_i} \mathrm{cf}\,\alpha = \aleph_0$.

Applying the induction to P_i (i is in B and so it is weakly compact) we obtain that P_i satisfies i-c.c., it does not add reals, and $i = \aleph_2^{V[\mathring{P}_i]}$.

The following lemma is proved in [5]:

Lemma 2.2. *$P^*[S_i]$ satisfies the strong $\mathbb{I}$-condition for a set $\mathbb{I}$ of monotone families so that* $\mathrm{NS}_{\aleph_2} \restriction S_i \in \mathbb{I}$, *where* $\mathrm{NS}_{\aleph_2} \restriction S_i = \{A \subseteq \aleph_2 \mid A \cap S_i$ *is a nonstationary subset of* $\aleph_2\}$.

By [13], the $\mathbb{I}$-condition and CH implies that $P^*[S_i]$ does not add reals. Let us show that $P^*[S_i]$ as the usual forcing for adding closed unbounded subsets does not add new functions from ω into On.

Lemma 2.3. *Every function $f \in V[\mathring{P}_{i+1}]$ from ω into $V[\mathring{P}_i]$ belongs to $V[\mathring{P}_i]$.*

Proof. Since $|P^*[S_i]| = \aleph_2$ in $V[P_i]$, it is enough to show that there is no such new f from ω into $i = \aleph_2^{V[P_i]}$.

Now $i \in B$ and so it is weakly compact. Hence the set

$$C = \{\alpha < i \mid \langle V_\alpha, \in, P_i \cap V_\alpha, S_i \cap \alpha, \mathbf{f} \cap V_\alpha\rangle < \langle V_i, \in, P_i, S_i, \mathbf{f}\rangle\}$$

contains a club, where $\mathbf{f}$ is a name of f in the forcing $P_i * P^*[S_i]$.

Let $\alpha \in S_i \cap C$ and $S_i \cap \alpha$ is stationary. Then α is an inaccessible. Hence $V_\alpha \cap P_i = P_\alpha$ and P_α satisfies α-c.c. Let $V_\alpha[\mathring{P}_\alpha]$ be $K''_{\mathring{P}_\alpha}(V_\alpha)$ (the interpretation of all the names which belong to V_α). Note that if $a \in V_\alpha$ then $K_{\mathring{P}_\alpha}(a) = K_{\mathring{P}_\beta}$ for some $\beta < \alpha$ since P_α satisfies α-c.c. So for every $a \in V_\alpha$, $K_{\mathring{P}_\alpha}(a) = K_{\mathring{P}_i}(a)$. Hence

$$\langle V_\alpha[\mathring{P}_\alpha], \in, P_\alpha, S_i \cap \alpha, \tilde{\mathbf{f}} \cap V_\alpha[\mathring{P}_\alpha]\rangle < \langle V_i[\mathring{P}_i], \in, P_i, S_i, \tilde{\mathbf{f}}\rangle,$$

where $\tilde{\mathbf{f}}$ is the interpretation of $\mathbf{f}$ in $V_i[\mathring{P}_i]$, i.e. $K''_{P_\alpha}(\mathbf{f})$.

In $V[\mathring{P}_{\alpha+1}]$, $\mathrm{cf}\,\alpha = \mathrm{cf}\,\alpha^+ = \aleph_0$, so we have a sequence $\langle \mathbf{C}_n \mid n < \omega\rangle$ such that

(a) $\mathbf{C}_n \in V$ and it is a club in α in $V[\mathring{P}_\alpha]$ (or in V; it does not matter since P_α satisfies α-c.c.),

(b) $\mathbf{C}_{n+1} \subseteq \mathbf{C}_n$,

(c) for every closed unbounded subset of α, $\mathbf{C} \in V[\mathring{P}_\alpha]$, there is some n so that $\mathbf{C}_n \subseteq \mathbf{C}$.

We take $q_0 \in V[P_\alpha]$ to be some $P^*[S_i]$-condition s.t.

$$q_0 \| \tilde{\mathbf{f}}(0).$$

Let $q_0^1 = q_0 \cup \{\alpha_0\}$, where $\alpha_0 \in \mathbf{C}_0 \cap S_i - q_0$. Find $q_1 \in V[P_\alpha]$ s.t. $q_1 \geqq q_0^1$ and $q_1 \| \tilde{\mathbf{f}}(1)$.

And so on.

Let $q = \bigcup q_n \cup \{\alpha\}$. Then $q \in P^*[S_i]$ and it forces that $\mathbf{f} \in V[\mathring{P}_i]$.

3. The idea

Let $A \subseteq (\kappa - B) \cap \{\alpha < \kappa \mid \alpha$ is a strongly inaccessible cardinal$\}$ and it belongs to $\mathcal{U}$, where B is from section 1. Then A is Wc_k-positive, and $A_\diamond = \{\alpha \in B \mid A \cap \alpha = S_\alpha$ and S_α is Wc_α-positive$\}$ is a stationary subset of κ. So for every $\alpha \in A_\diamond$ we forced with $P^*[S_\alpha]$. Hence in $V[\mathring{P}_\kappa]$, $\{\alpha < \kappa = \aleph_2 \mid \mathrm{cf}\, \alpha = \aleph_1$ and $A \cap \alpha$ contains an ω-closed unbounded subset of $\alpha\} \supseteq A_\diamond$ and $A_\diamond$ remains stationary in $V[\mathring{P}_\kappa]$ since P_κ satisfies κ-c.c. This is enough for shooting an ω-club through every $A \in \mathcal{U}$, without collapsing any cardinal. See [1]. The problem arises when we try to iterate such forcing.

In our case we don't need to be worried about every new subset of $\aleph_2$. The precipitousness will be preserved if we add to the filter (generated by $\mathcal{U}$) some special sets. Let us explain it more precisely. Let $A \in \mathcal{U}$ be as before. We force in $V[\mathring{P}_\kappa]$ with usual $P[A] = \{f \in V[\mathring{P}_\kappa] \mid f$ is an ω-closed subset of $A\}$. Let us define the extension $\mathcal{U}_0$ of $\mathcal{U}$ (in $V[\mathring{P}_\kappa * \mathring{P}[A]]$) as follows:

$E \in \mathcal{U}_0$ iff there is $\langle p, q\rangle \in \mathring{P}_\kappa * \mathring{P}[A]$ so that in the ultrapower N, $p \Vdash_{j(P_\kappa)}$ (for all $C' \subseteq P[A]$ which is generic over $N[\mathring{P}_\kappa]$ and $q \in C'$, $\bigcup C' \cup \{\kappa\} \Vdash_{P[j(A)]} \check{\kappa} \in j(\mathbf{E})$).

The direct way now is to shoot new ω-clubs through every $E \in \mathcal{U}_0$. But it is not clear why such forcing does not collapse $\aleph_2$.

Let us do something different from the direct shooting ω-clubs through elements of $\mathcal{U}_0$.

Suppose that $E \in \mathcal{U}_0$, then the set $E' = \{\alpha \in A \mid p \Vdash_{P_\kappa}$ (for all $C' \subseteq P[A \cap \alpha]$ which is generic over $V[\mathring{P}_\alpha]$ and $q \in C'$

$$\bigcup C' \cup \{\alpha\} \Vdash_{P[A_0]} \check{\alpha} \in \mathbf{E}\})$$

belongs to $\mathcal{U}$.

Let G be a $\langle V[\mathring{P}_\kappa], P[A]\rangle$-generic (we shall denote in such a way that $G \subseteq P[A]$ and it is a generic over $V[\mathring{P}_\kappa]$). We shall not distinguish between G and $\bigcup G$ which is the ω-closed subset of A.

Let $A' = \{\alpha \in A \mid G \cap \alpha$ is a $\langle V[\overset{\circ}{P}_\alpha], P[A \cap \alpha]\rangle$ generic$\}$. It is clear that $A' \cap E' \subseteq E$. $E' \in \mathcal{U}$ so we can shoot an ω-club through it without problems. Now if we succeed in doing this also with A', then E will contain an ω-club.

For such special sets A' we are ready to iterate our forcing. Let us define two such steps forcing and show that it does not collapse $\aleph_2$.

So let $A \in \mathcal{U}$ be as above. Let us define in $V[\overset{\circ}{P}_\kappa]$ the forcing notion $P^{(1)}[A]$ for shooting two ω-clubs, one through A and the second, which will be a subset of the first one, through the "generic points" of A.

$P^{(1)}[A]$ will be a set of all pairs $\langle c_0, c_1\rangle$ so that c_0, c_1 are ω-closed subsets of A, $c_0 \supseteq c_1$ and for every $\beta \in c_1$, $c_0 \cap \beta$ is a $\langle V[\overset{\circ}{P}_\beta], P[A \cap \beta]\rangle$-generic.

Let us show that this forcing does not collapse cardinals.

PROPOSITION 3.1. *The forcing $P^{(1)}[A]$ does not add new functions from $\aleph_1^{V[\overset{\circ}{P}_\kappa]}$ into $V[\overset{\circ}{P}_\kappa]$.*

PROOF. Let $A^{(1)} = \{\alpha \in A \mid A_\diamond \cap \alpha$ is a stationary subset of $\alpha\}$. Then $A^{(1)} \in \mathcal{U}$ since $A_\diamond$ is a stationary subset of κ in V and so also in N.

Now, as above, for every $\alpha \in A^{(1)}$ the forcing $P[A \cap \alpha]$ in $V[\overset{\circ}{P}_\alpha]$ does not collapse any cardinals. And more than that, we can find a generic subset of $P[A \cap \alpha]$ already in $V[\overset{\circ}{P}_{\alpha+1}]$. Since at the step α we forced with $\mathrm{Nm}'_{\alpha,\alpha^+}$, so $\mathrm{cf}\,\alpha = \mathrm{cf}(\alpha^+) = \aleph_0$ in $V[\overset{\circ}{P}_{\alpha+1}]$. Hence the set $\mathcal{D}$ of all dense subsets of $P[A \cap \alpha]$ which belongs to $V[\overset{\circ}{P}_\alpha]$ is of cardinality α^+ in $V[\overset{\circ}{P}_\alpha]$. So in $V[\overset{\circ}{P}_{\alpha+1}]$ $\mathcal{D} = \bigcup_{n<\omega} \mathcal{D}_n$, where each $\mathcal{D}_n \in V[\overset{\circ}{P}_\alpha]$ and it is of cardinality $\aleph_1$ in $V[\overset{\circ}{P}_\alpha]$. As in [3], by going through elementary submodels one can build a sequence $\langle q_n \mid n < \omega\rangle$ of elements of $P[A \cap \alpha]$ so that $q_{n+1} \geqq q_n$ and for every $T \in \mathcal{D}_n$, q_n is stronger than some element of T.

Now suppose that $\mathfrak{f}$ is a $P^{(1)}[A]$-name of a function from $\aleph_1^{V[\overset{\circ}{P}_\kappa]}$ into $\aleph_2^{V[\overset{\circ}{P}_\kappa]}$. As before, let us denote

$$A^{(1)}_\diamond = \{\alpha \in B \mid A^{(1)} \cap \alpha = S_\alpha \text{ and } S_\alpha \text{ is } \mathrm{Wc}_\alpha\text{-positive}\}.$$

It is a stationary subset of κ.

Let C be a club from Lemma 2.3 of elementary submodels of $\langle V_\kappa, \in, P_\kappa, A, A^{(1)}, \mathfrak{f}\rangle$.

Let $\alpha \in A^{(1)}_\diamond \cap C$. As in Lemma 2.3 then

$$\langle V_\alpha[\overset{\circ}{P}_\alpha], \in, P_\alpha, A \cap \alpha, S_\alpha, \tilde{\mathfrak{f}} \cap V_\alpha[\overset{\circ}{P}_\alpha]\rangle < \langle V_\kappa[\overset{\circ}{P}_\kappa], \in, P_\kappa, A, A^{(1)}, \tilde{\mathfrak{f}}\rangle.$$

Note that since α is an inaccessible, it is a limit point of C since we can consider elementary submodels of $\langle V_\alpha, \in, P_\alpha, A \cap \alpha, S_\alpha, \mathfrak{f} \cap V_\alpha\rangle$.

We forced with $P^*[S_\alpha]$ on step α. So let G_α be a generic subset of $P^*[S_\alpha]$ and belonging to $V[P_{\alpha+1}]$. The cofinality of α in $V[P_{\alpha+1}]$ is $\aleph_1$ (by Lemma 2.3). Let $E = \bigcup G_\alpha \cap C$. Then it is a closed unbounded subset of α. Fix some increasing continuous enumeration $\langle \mu_\nu \mid \nu < \mathrm{cf}_\alpha^{V[\mathring{P}_{\alpha+1}]} \rangle$ of it. Since every member of S_α is an inaccessible cardinal in V, we can apply to it the argument from Lemma 2.3 and obtain that

$$\begin{aligned} N_\nu &= \langle V_{\mu_\nu}[\mathring{P}_{\mu_\nu}], \in, P_{\mu_\nu}, A \cap \mu_\nu, S_\alpha \cap \mu_\nu, \tilde{\mathbf{f}} \cap V[\mathring{P}_{\mu_\nu}] \rangle \\ &< \langle V_\alpha[\breve{P}_\alpha], \in, P_\alpha, A \cap \mu_\nu, S_\alpha, \tilde{\mathbf{f}} \cap V[\mathring{P}_\alpha] \rangle. \end{aligned}$$

Since E is a club in α it implies that the last model is the union of the elementary chain $\langle N_\nu \mid \nu < \mathrm{cf}\, \alpha^{V[\breve{P}_{\alpha+1}]} \rangle$.

Now let us define in $V[\breve{P}_{\alpha+1}]$ a sequence $\langle q_\nu \mid \nu < \aleph_1 \rangle$ so that

(i) $q_\nu \in P^{(1)}[A \cap \mu_\nu] \cap V[\breve{P}_{\mu_\nu+1}]$,

(ii) $q_\nu = \langle c_{0\nu}, c_{1\nu} \rangle$ and $\max c_{i\nu} = \mu_\nu$ for $i = 0, 1$,

(iii) $q_{\nu+1}$ decides $\mathbf{f}(\nu)$,

(iv) $q_{\nu+1} \geqq q_\nu$.

Since every μ_ν belongs to $A^{(1)}$, as we explained above, we can define q_ν on nonlimit stage ν. Inside $N_{\nu+1}$ find some $q'_\nu \geqq q_\nu$ which decides $f(\nu)$ and let $q_{\nu+1}$ be an element stronger than q'_ν which satisfies (ii).

For limit ν let $q_\nu = \langle c_{0\nu}, c_{1\nu} \rangle$ where $c_{i\nu} = \bigcup_{\nu' < \nu} c_{i\nu'} \cup \{\mu_\nu\}$ for $i = 0, 1$.

Let us prove that $q_\nu \in P^{(1)}[A]$. Note that it is enough, since the sequence $\langle \mu_{\nu'} \mid \nu' < \nu \rangle$ is a countable subset of μ_ν, $|\mu_\nu|^{V[\mathring{P}_{\mu_\nu+1}]} = \aleph_1$ and since P_κ does not add reals the forcing $P_\kappa / \breve{P}_{\mu_\nu+1}$ does not add new ω-sequences to μ_ν. Hence $\langle \mu_{\nu'} \mid \nu' < \nu \rangle \in V[\mathring{P}_{\mu_\nu+1}]$.

Let us prove that $c_{0\nu} \cap \mu_\nu$ is a $\langle V[P_{\mu_\nu}], P[A \cap \mu_\nu] \rangle$-generic. So let $D \in V[P_{\mu_\nu}]$ be a dense subset of $P[A \cap \mu_\nu]$. Note that $P[A \cap \mu_\nu] \subseteq V_{\mu_\nu}[\mathring{P}_{\mu_\nu}]$. So let $\mathbf{D}$ be a name of D which is a subset of V_{μ_ν}. Now let us consider

$$R = \{\alpha < \mu_\nu \mid \langle V_\alpha, \in, P_\alpha, A \cap \alpha, \mathbf{D} \cap V_\alpha \rangle < \langle V_{\mu_\nu}, \in, P_{\mu_\nu}, A \cap \mu_\nu, \mathbf{D} \rangle \}.$$

Then R is a closed unbounded subset of μ_ν in V, since μ_ν is an inaccessible there.

Remember that μ_ν is a limit point of $\bigcup G_\alpha$ (a generic subset of $P^*[S_\alpha]$), so $(\bigcup G_\alpha) \cap \mu_\nu$ intersects every closed unbounded subset of μ_ν in $V[P_{\mu_\nu}]$. Hence there is $\mu \in \bigcup G_\alpha \cap \mu_\nu \cap (C \cap R)$ (μ_ν is also a limit point of C, so $C \cap \mu_\nu$ is a club in V). Then

$$\langle V_\mu[\mathring{P}_\mu], \in, P_\mu, A \cap \mu, D \cap V_\mu[\mathring{P}_\mu] \rangle < \langle V_{\mu_\nu}[\mathring{P}_{\mu_\nu}], \in, P_{\mu_\nu}, A \cap \mu_\nu, D \rangle$$

and so $D \cap V_\mu[\mathring{P}_\mu]$ is a dense subset of $P[A \cap \mu]$. Now $\mu = \mu_{\nu_1}$ for some $\nu_1 < \nu$

and $q_{\nu_1} \in P^{(1)}[A \cap \mu]$, $q_{\nu_1} = \langle c_{0\nu_1}, c_{1\nu_1} \rangle$ and $\max c_{i\nu_1} = \mu$ for $i = 0, 1$, so $c_{0\nu_1}$ is stronger than some element of $D \cap V_\mu[\mathring{P}_\mu]$. But $c_{0\nu} \geqq c_{0\nu_1}$, hence $c_{0\nu}$ also is stronger than some element of D.

So we proved that $c_{0\nu} \cap \mu_\nu$ is a $\langle V[\mathring{P}_\mu], P[A \cap \mu_\nu]\rangle$-generic. It implies that $q_\nu \in P^{(1)}[A]$. $\square$

Let $A^{(2)} = \{\alpha \in A^{(1)} \mid A^{(1)}_\diamond \cap \alpha$ is a stationary subset of $\alpha\}$. Then $A^{(2)} \in \mathcal{U}$ and using the ideas from Proposition 3.1, we can show that for every $\alpha \in A^{(2)}$ the forcing $P^{(1)}[A \cap \alpha]$ in $V[P_\alpha]$ does not collapse any cardinals and in $V[\mathring{P}_{\alpha+1}]$ there is a $\langle V[P_\alpha], P^{(1)}[A \cap \alpha]\rangle$-generic set.

Let $P^{(2)}[A] = \{\langle c_0, c_1, c_2\rangle \mid c_0, c_1, c_2$ are ω-closed subsets of A, $c_0 \supseteq c_1 \supseteq c_2$, for every $\beta \in c_2$, $\langle c_0 \cap \beta, c_1 \cap \beta\rangle$ is a $\langle V[\mathring{P}_\beta], P^{(1)}[A]\rangle$-generic$\}$.

In the same way we define $A^{(n)}$ and $P^{(n)}[A]$ for $n < \omega$. Let $A^{(\omega)} = \bigcap_{n<\omega} A^{(n)}$ and $P^{(\omega)}[A]$ be the set of all sequences $\langle c_0, \ldots, c_n, \ldots \mid n < \omega\rangle$ so that for every n, $\langle c_0, \ldots, c_n\rangle \in P^{(n)}[A]$. Now why does $P^{(\omega)}[A]$ not collapse cardinals? The idea is as in Proposition 3.1. Instead of $A^{(1)}$ we take

$$A^{(\omega+1)} = \mathrm{df}\, (A^{(\omega)})^{(1)} = \{\alpha \in A^{(\omega)} \mid A^{(\omega)}_\diamond \cap \alpha \text{ is stationary}\}.$$

Also we prove that for every $\alpha \in A^{(\omega)}$ there is $\langle c_n \mid n < \omega\rangle \in P^{(\omega)}[A]$ so that for every n, $\bigcup c_n = \alpha$ and $\langle c_0, \ldots, c_n\rangle$ is $\langle V[\mathring{P}_\alpha], P^{(n)}[A]\rangle$-generic. We shall not give the proof here. It will be done in the next section in a general situation.

It is possible to continue and define $A^{(\alpha)}$ and $P^{(\alpha)}[A]$ for every $\alpha < \kappa^+$. For α of cofinality κ the definition of $A^{(\alpha)}$ uses a diagonal intersection. It can be done in such a way that $P^{(\alpha)}[A]$ satisfies κ^+-c.c., namely let $P^{(\alpha)}[A] = \bigcup_{\beta<\alpha} P^{(\beta)}[A]$ for α of cofinality κ. We define $P^{(\kappa^+)}[A] = \bigcup_{\alpha<\kappa^+} P^{(\alpha)}[A]$. Also $P^{(\kappa^+)}[A]$ will satisfy κ^+-c.c. and will not collapse cardinals. Now every new set, which must be included into the filter generated by $\mathcal{U}$, appears at some stage $\alpha < \kappa^+$. Already at the next stage $\alpha + 1$, after we force with $P^{(\alpha+1)}[A]$ it will contain some set $A_1 \in \mathcal{U}$ intersected with an ω-closed unbounded set.

This is the idea. In the next section we shall define and force with this kind of forcing but at the same time for every $A \in \mathcal{U}$.

4. The main forcing

Fix some enumeration of the set $\{A \in \mathcal{U} \mid A \subseteq \kappa - B$ and every $\alpha \in A$ is an inaccessible$\}$ by nonlimit ordinals $\langle A_{\nu+1} \mid \nu < \kappa^+\rangle$. For a limit ν let us define an element A_ν of $\mathcal{U}$ in a special way.

First let us define A_ν for limit $\nu < \kappa$. Let $\bar{A}_\nu = \bigcap_{\mu<\nu} A_\mu$. Now let $A_\nu = \bar{A}^{(2)}_\nu$, where as in Section 3 for A in $\mathcal{U}$ we denote by $A^{(1)}$ the set of all $\alpha \in A$ so that

$A_\diamond \cap \alpha$ is stationary in α (i.e., A is guessed below α stationary many times), and $A^{(2)} = (A^{(1)})^{(1)}$.

For ν, $\kappa^+ > \nu \geqq \kappa$, we shall build some diagonal intersection. First, fix for every $\nu \leqq \kappa^+$ a cofinal increasing continuous sequence $\langle \nu_\tau \mid \tau < \operatorname{cf} \nu \rangle$, so that if there is $\mu < \nu$, μ limit and $\nu = \mu + \omega$, then let $\nu_0 = \mu$ and $\nu_n = \mu + n$ for $n < \omega$, otherwise every ν_τ is a limit ordinal.

Now we define $\bar{A}_\nu = \bigcap_{\tau < \operatorname{cf} \nu} A_{\nu_\tau}$ if $\operatorname{cf} \nu < \kappa$ and $\bar{A}_\nu = \triangle_{\tau < \kappa} A_{\nu_\tau} = \{\beta < \kappa \mid \forall \tau < \beta, \beta \in A_{\nu_\tau}\}$ if $\operatorname{cf} \nu = \kappa$. As above let $A_\nu = \bar{A}_\nu^{(2)}$.

For $\alpha < \kappa^+$ let us fix some $i_\alpha : \kappa \to \alpha$ so that

(i) if $\alpha < \kappa$, $i_\alpha(\beta) = \beta$, for $\beta < \alpha$ and $i_\alpha(\beta) = 0$, otherwise;

(ii) if $\alpha = \kappa$, i_α is the identity function:

(iii) if $\kappa < \alpha < \kappa^+$, i_α is a 1-1 mapping from κ onto α.

For $\alpha < \kappa^+$ let us define a closed unbounded subset of α, C_α, so that its elements will be closed enough under i_α. Let $C_\alpha = \kappa - \alpha$ for $\alpha < \kappa$ and $C_\kappa = \kappa$. For $\alpha > \kappa$ let us consider first the structure

$$\mathscr{A}_{\kappa,\alpha} = \langle \alpha, \in, i_\alpha, \kappa, R_0, \langle \alpha_\tau \mid \tau \langle \operatorname{cf} \alpha \rangle, R_1 \rangle,$$

where

$$R_0 = \{\langle \delta, \tau, \delta_\tau \rangle \mid \delta < \alpha, \tau < \kappa \text{ and } \langle \delta_\tau \mid \tau < \kappa \rangle \text{ is the picked cofinal sequence to } \delta\},$$

$$R_1 = \{\langle \delta, \tau, \mu \rangle \mid \delta < \alpha, \tau < \kappa \text{ and } i_\delta(\tau) = \mu\}\rangle.$$

Let now $W < \mathscr{A}_{\kappa,\alpha}$ and $|W| < \kappa$. Suppose also that $W \cap \kappa$ is some ordinal β. Then W is equal to $\mathscr{A}_{\beta,\alpha} =_{\mathrm{df}} \langle i''_\alpha(\beta), \in, i_\alpha \restriction \beta, \kappa, R_0 \restriction \beta, \quad \langle \alpha_\tau \mid \tau < \min(\operatorname{cf} \alpha, \beta) \rangle, R_1 \restriction \beta \rangle$, $R_0 \restriction \beta =_{\mathrm{df}} \{\langle \delta, \tau, \delta_\tau \rangle \mid \delta \in i''_\alpha(\beta),\ \tau < \beta$ and δ_τ is from $\langle \delta_\tau \mid \tau < \operatorname{cf} \delta \rangle\}$, $R_1 \restriction \beta =_{\mathrm{df}} \{\langle \delta, \tau, \mu \rangle \mid \delta \in i''_\alpha(\beta),\ \tau < \beta$ and $i_\delta(\tau) = \mu\}$.

Since κ is an inaccessible, $\bar{C}_\alpha = \{\beta < \kappa \mid \mathscr{A}_{\beta,\alpha} < \mathscr{A}_{\kappa,\alpha}\}$ contains a club. Let γ be the least ordinal s.t. $\gamma \geqq \operatorname{cf}^V \alpha$ and $\operatorname{cf}^{V[\check{P}_\gamma]}(\alpha) = \operatorname{cf}^{V[\check{P}_\kappa]} \alpha$ if $\operatorname{cf} \alpha < \kappa$ and 0 if $\operatorname{cf} \alpha = \kappa$.

Now put $C_\alpha = \{\beta < \kappa \mid \beta > \gamma$ and β is a limit point of $\bar{C}_\alpha\}$. Note that every inaccessible cardinal $\beta > \gamma$ in $\bar{C}_\alpha$ is a limit of $\bar{C}_\alpha$ point and so belongs to C_α.

LEMMA 4.1. *Let $\kappa \leqq \alpha_1 < \alpha_2$, $\beta < \kappa$ be so that $\beta \in \bar{C}_{\alpha_2}$ and $\alpha_1 \in i''_{\alpha_2}(\beta)$, then $\beta \in \bar{C}_{\alpha_1}$.*

PROOF. It is enough to show that $\mathscr{A}_{\beta,\alpha_1} < \mathscr{A}_{\kappa,\alpha_1}$. But since $\alpha_1 \in i''_{\alpha_2}(\beta)$, $i_{\alpha_1} \mid \beta$ is definable in $\mathscr{A}_{\beta,\alpha_2}$. For a formula $\varphi(\tau_1, \ldots, \tau_n)$ where $\tau_1, \ldots, \tau_n \in i''_{\alpha_1}(\beta)$ let $\varphi^{i''_{\alpha_1}(\beta)}(\tau_1, \ldots, \tau_n)$ be the formula obtained from φ by the restriction of all the quantifiers to $i''_{\alpha_1}(\beta)$ (i.e. for $\exists x \Psi$, $(\exists x \Psi)^{i''_{\alpha_1}(\beta)}$ is $\exists x \in i''_{\alpha_1}(\beta) \Psi^{i_{\alpha_1}(\beta)}$ and so on).

Then $$\mathscr{A}_{\beta,\alpha_1} \models \varphi(\tau_1, \ldots, \tau_n)$$

iff $$\mathcal{A}_{\beta,\alpha_1} \models \varphi^{i''_{\alpha_1}(\beta)}(\tau_1, \ldots, \tau_n)$$

iff $$\mathcal{A}_{\kappa,\alpha_2} \models \varphi^{\alpha_1}(\tau_1, \ldots, \tau_n)$$

iff $$\mathcal{A}_{\kappa,\alpha_1} \models \varphi(\tau_1, \ldots, \tau_n).$$

Lemma 4.2. *Let $\kappa \leqq \alpha_1 \leqq \alpha_2$, $\beta < \kappa$ be so that $\beta \in C_{\alpha_2}$ and $\alpha_1 \in i''_{\alpha_2}(\beta)$ then $\beta \in C_{\alpha_1}$.*

Proof. By the definition of C_{α_1}, it is enough to show that β is a limit point of $\bar{C}_{\alpha_1}$. Now, since β is a limit point of $\bar{C}_{\alpha_2}$ and $\alpha_1 \in i''_{\alpha_2}(\beta)$, there are unboundedly many in $\beta, \delta \in \bar{C}_{\alpha_2}$ s.t. $\alpha_1 \in i''_{\alpha_2}(\delta)$. By Lemma 4.1, every such δ belongs to $\bar{C}_{\alpha_1}$ and, also, $\beta \in \bar{C}_{\alpha_1}$. Hence β is a limit point of $\bar{C}_{\alpha_1}$. $\square$

Main Definition. For $\nu < \kappa^+$ we define in $V[\breve{P}_\kappa]$ by induction the forcing notion Q_ν and the ordering $\leqq_\nu$ on it as follows:

An element $q \in Q_\nu$ is a sequence $\{\langle \alpha, q_\alpha \rangle \mid \alpha \in i''_\nu(\beta_q)\}$, where β_q is some element of C_ν, so that:

(1) For every $\alpha \in i''_\nu(\beta_q)$, q_α is an ω-closed subset of A_α of cardinality less than $\aleph_2$.

(2) For every limit $\alpha \in i''_\nu(\beta_q)$, q_α is a subset of C_α and, if $\beta \in q_\alpha$, then $i''_\alpha(\beta) \subseteq i''_\nu(\beta_q)$, $\beta \in q_\tau$ for every $\tau \in i''_\alpha(\beta)$ and

$$q \restriction \langle \alpha, \beta \rangle =_{\mathrm{df}} \{\langle \tau, q_\tau \cap \beta \rangle \mid \tau \in i''_\alpha(\beta)\}$$

is a $\langle V[\breve{P}_\beta], Q_\alpha \restriction \beta \rangle$-generic, where $Q_\alpha \restriction \beta =_{\mathrm{df}} \{p \in Q_\alpha \cap V[\breve{P}_\beta] \mid \beta_p < \beta$ and for every $\tau \in i''_\alpha(\beta_p)$, p_τ is bounded in $\beta\}$.

For $p, q \in Q_\nu$ we define $p \geqq_\nu q$ (p is stronger than q) if $\beta_p \geqq \beta_q$ and for every $\alpha \in i''_\nu(\beta_q)$, p_α is an end extension of q_α.

Remark. (i) For α, β as in (2) if $\alpha \geqq \kappa$ then $\alpha \in i''_\nu(\beta_q)$ implies by Lemma 4.2 $\beta_q \in C_\alpha$. Since i_α is a 1-1 function $i''_\alpha(\beta) \not\subseteq i''_\alpha(\beta_q) = i''_\nu(\beta_q) \cap \alpha$ if $\beta > \beta_q$. So $\beta_q \geqq \beta$. $i''_\alpha(\beta_q) \subseteq i''_\nu(\beta_q) \cap \alpha$, since by the definition of $\mathcal{A}_{\beta_q,\nu}$, $i''_\nu(\beta_q)$ is closed under $i_\alpha \restriction \beta_q$. Now, for every $\tau_1 \in i''_\nu(\beta_q) \cap \alpha$ there is $\tau_2 < \beta_q$ s.t. $\tau_1 = i_\alpha(\tau_2)$, since $\mathcal{A}_{\beta_q,\nu} < \mathcal{A}_{\kappa,\nu}$. Hence $i''_\alpha(\beta_q) = i''_\nu(\beta_q) \cap \alpha$.

(ii) $Q_\alpha \restriction \beta$ can be defined inside $V[\breve{P}_\beta]$. Hence $Q_\alpha \restriction \beta \in V[\breve{P}_\beta]$.

(iii) For every ν, $|Q_\nu| = \kappa = \aleph_2^{V[\breve{P}_\kappa]}$.

Definition 4.3. Let $Q_{\kappa^+} = \bigcup_{\nu < \kappa^+} Q_\nu$ and for $p, q \in Q_{\kappa^+}$ let $p \geqq q$ if for every α s.t. q_α is defined and is not the empty set, p_α is an end extension of it.

We would like to show that for every $\nu < \kappa^+$, $Q_\nu \lessdot Q_{\kappa^+}$, i.e., every maximal antichain of Q_ν is a maximal antichain of Q (hence compatibility is preserved). It is clear that if $p \geqq_\nu q$ then $p \geqq q$.

Lemma 4.4. *$p_i \in Q_{\nu_i}$ $(i \in 2)$ are incompatible in Q_{κ^+} iff for some $\alpha \in i''_{\nu_1}(\beta_{p_1}) \cap i''_{\nu_2}(\beta_{p_2})$, for every $i, j \in 2$, $i \neq j$, p_{i_α} is not an end extension of p_{j_α}.*

Proof. (1) $\Leftarrow$ By Definition 4.3.

(2) $\Rightarrow$ Suppose that for every $\alpha \in i_{\nu_1}(\beta_{p_1}) \cap i''_{\nu_2}(\beta_{p_2})$, p_{i_α} is an end extension of p_{j_α} for some $i \neq j$, $i, j \in 2$. Let $\nu_1 \leqq \nu_2$. We define $p' = \{\langle \alpha, p'_\alpha \rangle \mid \alpha \in i''_{\nu_1}(\beta)\}$, where β is some element of $C_{\nu_2} - (\beta_{p_1} \cup \beta_{p_2})$ so that $\nu_1 \in i''_{\nu_2}(\beta)$, $p'_\alpha = p_{1\alpha}$ if $\alpha \in i''_{\nu_1}(\beta_{p_1})$ and $p'_\alpha = \emptyset$ otherwise. Note that such defined $p' \in Q_{\nu_1}$ and $p' \geqq_{\nu_1} p_1$. Let $p'' = \{\langle \alpha, p''_\alpha \rangle \mid \alpha \in i''_{\nu_2}(\beta)\}$, where $p''_\alpha = p'_\alpha$ if $\alpha \in i''_{\nu_1}(\beta)$ and $p''_\alpha = \emptyset$ otherwise. Also $p'' \in Q_{\nu_2}$. Let us call such kinds of extensions, trivial extensions. Clearly, p'' is stronger, in the ordering of Q_{κ^+}, than p_1. Let us find some $q \in Q_{\nu_2}$, $q \geqq_{\nu_2} p''$ and $q \geqq_{\nu_2} p_2$.

By taking some trivial extension p'_2 of p_2 we can make $\beta_{p'_2} = \beta$. So assume that already $\beta_{p_2} = \beta$. Now let us define $q = \{\langle \alpha, p'_\alpha \cup p_{2\alpha} \rangle \mid \alpha \in i''_{\nu_2}(\beta)\}$. It is enough to show that $q \in Q_{\nu_2}$ and then, obviously, $q \geqq_{\nu_2} p''$, p_2. So let us check the condition (2) from the definition of Q_{ν_2}. Let a limit ordinal $\alpha \in i''_{\nu_2}(\beta)$ and $\beta \in q_\alpha = p'_\alpha \cup p_{2\alpha}$. Now $q_\alpha = p'_\alpha$ or $q_\alpha = p_{2\alpha}$. Suppose $q_\alpha = p'_\alpha$ (the case $q_\alpha = p_{2\alpha}$ is the same). Then $p'' \restriction \langle \alpha, \beta \rangle$ is a $\langle V[\mathring{P}_\beta], Q_\alpha \restriction \beta \rangle$-generic. But $p'' \restriction \langle \alpha, \beta \rangle = q \restriction \langle \alpha, \beta \rangle$. Since for every $\tau \in i''_\alpha(\beta)$, $\beta \in p''_\tau$ and hence, since if $q_\tau \neq p''_\tau$, then $p_{2\tau}$ is an end extension of p''_τ, so $p''_\tau \cap \beta = p_{2\tau} \cap \beta = q_\tau \cap \beta$. □

Lemma 4.5. *For every $\nu < \kappa^+$*

(i) *$Q_\nu \lessdot Q_{\kappa^+}$, i.e. every maximal antichain of Q_ν is a maximal antichain of Q_{κ^+}.*

(ii) *If G is a generic subset of Q_{κ^+}, then $G \cap Q_\nu$ is a generic subset of Q_ν.*

Proof. Clearly (i) implies (ii). So let us prove (i). Suppose that $\langle p^\mu \mid \mu < \lambda \rangle$ is a maximal antichain in Q_ν. Let $p \in Q_\alpha$ for some $\alpha < \kappa^+$. Suppose that p is incompatible in Q_{κ^+} with every p^μ $(\mu < \lambda)$. By taking the trivial extensions of p, we can make $\alpha \geqq \nu$ and $\nu \in i''_\alpha(\beta_p)$. Let us consider $p \restriction \nu =_{\mathrm{df}} \{\langle \tau, p_\tau \rangle \mid i''_\nu(\beta_p)\}$. Then $p \restriction \nu \in Q_\nu$ since $\nu \in i''_\alpha(\beta_p)$ implies $i''_\alpha(\beta_p) \supseteq i''_\nu(\beta_p)$, for $\beta_p \in C_\alpha$. Also $p \restriction \nu \leqq p$ in Q_{κ^+}. Now for some $\mu < \lambda$, p^μ is compatible in Q_ν with $p \restriction \nu$. Let $q \in Q_\nu$, $q_\nu \geqq p^\mu$, $p \restriction \nu$. We assumed that p and q are incompatible in Q_{κ^+}. So by Lemma 4.4, for some $\gamma \in i''_\nu(\beta_q) \cap i''_\alpha(\beta_p)$, q_γ is not an end extension of p_γ or the converse. But $\mathscr{A}_{\alpha,\beta_p} < \mathscr{A}_{\alpha,\kappa}$ and $\mathscr{A}_{\alpha,\kappa} \models \exists \tau < \kappa\ \gamma = i_\nu(\tau)$. Hence there is $\tau < \beta_p$, $\gamma = i_\nu(\tau)$. So $\gamma \in i''_\nu(\beta_p)$ and p_γ is in $p \restriction \nu$, which is impossible. Contradiction.

Lemma 4.6. *Q_{κ^+} satisfies κ^+-c.c.*

PROOF. Suppose that T is a maximal antichain in Q_{κ^+}. Since for every $\nu < \kappa^+$, $|Q_\nu| = \kappa$, we can find $\mu < \kappa^+$, $\mathrm{cf}\,\mu = \kappa$, so that for every $q \in \bigcup_{\nu<\mu} Q_\nu$ there is $t \in T \cap \bigcup_{\nu<\mu} Q_\nu$ compatible with q.

Let us show that then $T \subseteq \bigcup_{\nu<\mu} Q_\nu$. Suppose otherwise. Then there is some $t \in T - \bigcup_{\nu<\mu} Q_\nu$. Let $t \in Q_\alpha$, $\alpha \geqq \mu$. As above, w.l.o.g. we can assume $\mu \in i''_\alpha(\beta_t)$. As in Lemma 4.5, then $t \restriction \mu \in Q_\mu$ and $t \restriction \mu \leqq t$ in the ordering of Q_{κ^+}. Let us find some inaccessible $\beta > \beta_t$ in C_μ. Since the cofinal sequence to μ, $\langle \mu_\tau \mid \tau < \kappa \rangle$ is in $\mathscr{A}_{\kappa,\mu}$, $\langle \mu_\tau \mid \tau < \beta \rangle$ represents it in $\mathscr{A}_{\beta,\mu}$. So $i''_\mu(\beta) = \bigcup_{\tau<\beta} i''_{\mu_\tau}(\beta)$ and for $\tau_1 > \tau_2$, $i''_{\mu_{\tau_1}}(\beta) \supseteq i''_{\mu_{\tau_2}}(\beta)$. Since $\mathscr{A}_{\kappa,\mu} \vDash \mu_{\tau_1} = i''_{\mu_{\tau_1}}(\kappa) \supseteq \mu_{\tau_2} = i''_{\mu_{\tau_2}}(\kappa)$. The cardinality of $i''_\mu(\beta_t)$ is $|\beta_t| < \beta$ (in V). So for some $\bar{\tau} < \beta$, $i''_{\mu_{\bar\tau}}(\beta) \supseteq i''_\mu(\beta_t)$. Let us consider $s = \{\langle \gamma, s_\gamma \rangle \mid \gamma \in i''_{\mu_{\bar\tau}}(\beta)\}$ where $s_\gamma = t_\gamma$ for $\gamma \in i''_\mu(\beta_t)$ and $s_\gamma = \varnothing$ otherwise. As above $s \in Q_{\mu_{\bar\tau}}$ and $s \geqq t \restriction \mu$. Now there is $t_1 \in T \cap \bigcup_{\nu<\mu} Q_\nu$ which is compatible with s. But hence it is compatible with $t \restriction \mu$ and with t, by Lemma 4.4. Contradiction.

5. The cardinals are preserved

First we are going to prove the following.

PROPOSITION 5.1. *For any limit ordinal $\nu < \kappa^+$, an ordinal $\alpha \in A_\nu \cap C_\nu$ and $p \in Q_\nu \restriction \alpha$, in the model $V[\mathring{P}_{\alpha+1}]$ there is a $\langle V[\mathring{P}_\alpha], Q_\nu \restriction \alpha \rangle$-generic set $q = \{\langle \tau, q_\tau \rangle \mid \tau \in i''_\nu(\alpha)\}$ so that $q \in Q_\nu$, $q \geqq_\nu p$ and $\bigcup (q_\tau \cap \alpha) = \alpha$ for every $\tau \in i''_\nu(\alpha)$.*

REMARK. We do not distinguish between a generic subset $G \subseteq Q_\nu \restriction \alpha$ and the set which we obtain from it by taking the union of the second coordinates of its elements, and also we add to each of the second coordinates its sup.

PROOF. We shall prove this proposition by induction. Suppose it is proved for every $\langle \mu, \beta \rangle$ s.t. μ is a limit ordinal $< \nu$ and $\beta \in A_\mu \cap C_\mu$, or $\mu = \nu$ and β is less than α.

Let us show first that the following holds:

LEMMA 5.2. *Let $\nu < \kappa^+$ be a limit ordinal and $\alpha \in \bar{A}^{(1)}_\nu \cap C_\nu$ (where $\bar{A}_\nu$ is from the definition of the $\langle A_\beta \mid \beta < \kappa^+ \rangle$). Then for every $p \in Q_\nu \restriction \alpha$ there is $q = \{\langle \tau, q_\tau \rangle \mid \tau \in i''_\nu(\alpha)\} \in Q_\nu \cap V[\mathring{P}_{\alpha+1}]$ so that $q \geqq_\nu p$* and for every $\tau \in i''_\nu(\alpha)$, $\alpha = \max q_\tau$.

REMARK. Note that if $\alpha \in \bar{A}_\nu \cap C_\nu$, then for every $\tau \in i''_\nu(\alpha)$, $\alpha \in A_\tau \cap C_\tau$. Let us prove it by induction on ν. If $\nu = \mu + \omega$, for some limit μ, then

$$\bar{A}_\nu = A_\mu \cap A_{\mu+1} \cap \cdots \cap A_{\mu+n} \cap \cdots$$

and

$$i''_\nu(\alpha) = \bigcup_{n<\omega} i''_{\mu+n}(\alpha) = i''_\mu(\alpha) \cup \{\mu + n \mid n < \omega\}.$$

If $\nu \neq \mu + \omega$ for any μ and $\mathrm{cf}^V \nu < \kappa$, then $\bar{A}_\nu = \bigcap \{A_{\nu_\delta} \mid \delta < \mathrm{cf}^V \nu\}$. So $\tau \in i''_{\nu_\delta}(\alpha)$ for some $\delta < \mathrm{cf}^V \nu$ and $\alpha \in A_{\nu_\delta} \cap C_{\nu_\delta}$. In the last case, when $\mathrm{cf}^V \nu = \kappa$, $\bar{A}_\nu = \Delta\{A_{\nu_\delta} \mid \delta < \kappa\}$. Since $\alpha \in C_\nu$, $i''_\nu(\alpha) = \bigcup \{i''_{\nu_\delta}(\alpha) \mid \delta < \alpha\}$. Also $\alpha \in A_{\nu_\delta} \cap C_{\nu_\delta}$ for every $\delta < \alpha$.

Proof. We shall consider four cases.

Case 1. There is the maximal limit $\mu < \nu$.

Then $\nu = \mu + \omega$ and since $\alpha \in C_\nu$, $\mu \in i''_\nu(\alpha)$ and $i''_\nu(\alpha) = i''_\nu(\alpha) \cup \{\mu + n \mid n < \omega\}$. So if $\mu_1 \in i''_\nu(\alpha)$ and it is a limit ordinal, then $\mu_1 \in i''_\mu(\alpha)$.

Now let $p \in Q_\nu \restriction \alpha$. Then $p = \{\langle \tau, p_\tau \rangle \mid \tau \in i''_\mu(\beta_p)\} \cup \{\langle \mu + n, p_{\mu+n} \rangle \mid n < \omega\}$.

α belongs to $A_\mu \cap C_\mu$. So we can apply the inductive hypothesis to $\langle \mu, \alpha \rangle$ and $p \restriction \mu$. Let $t \in Q_\mu \cap V[\mathring{P}_{\alpha+1}]$ be as it claims. Let us define

$$q = t \cup \{\langle \mu + n, p_{\mu+n} \cup \{\alpha\} \rangle \mid n < \omega\}.$$

Then $q \in Q_\nu$ since every limit $\mu_1 \in i''_\nu(\alpha)$ is equal to μ, or belongs to $i''_\mu(\alpha)$. In case $\beta \in p_\mu$, we have that $p \restriction \langle \mu, \beta \rangle$ is $\langle V[\mathring{P}_\beta], Q_\mu \restriction \beta \rangle$-generic. Also for every $\tau \in i''_\mu(\beta)$, $\beta \in p_\tau$, so $t_\tau \cap \beta = p_\tau \cap \beta$ and hence $p \restriction \langle \mu, \beta \rangle = t \restriction \langle \mu, \beta \rangle$. Also $q \geqq_\nu p$ since $t \geqq_\mu p \restriction \mu$.

Case 2. $\mathrm{cf}^{V[\mathring{P}_\kappa]} \nu = \aleph_0$.

Since $\alpha \in C_\nu$, $\alpha > \mathrm{cf}^V \nu$. So all the cofinal sequence to ν is contained in $i''_\nu(\alpha)$.

Let us pick in $V[\mathring{P}_\alpha]$ a sequence of limit ordinals $\nu_0 < \nu_1 < \cdots < \nu_n < \cdots$ cofinal in ν, from the elements of the old sequence to ν. Then

$$\{\nu_n \mid n < \omega\} \subseteq i''_\nu(\alpha), \quad i''_\nu(\alpha) = \bigcup_{n<\omega} i''_{\nu_n}(\alpha) \quad \text{and} \quad \nu_n \in i''_{\nu_{n+1}}(\alpha).$$

Suppose now that $p = \{\langle \tau, p_\tau \rangle \mid \tau \in i''_\nu(\beta_p)\} \in Q_\nu \restriction \alpha$. Then $\beta_p \in C_\nu$ and so $\beta_p > \mathrm{cf}^V \nu$, $\{\nu_n \mid n < \omega\} \subseteq i''_\nu(\beta_p)$, $i''_\nu(\beta_p) = \bigcup_{n<\omega} i''_{\nu_n}(\beta_p)$ and $\nu_n \in i''_{\nu_{n+1}}(\beta_p)$. Let us denote $p \restriction \nu_n$ by p_n. Then $p_n \in Q_{\nu_n} \restriction \alpha$ and $p = \bigcup_{n<\omega} p_n$.

Since $\alpha \in A_\nu$, we forced on the step α with $\mathrm{Nm}'_{\alpha,\alpha^+}$. So in $V[\mathring{P}_{\alpha+1}]$ there is a sequence $\langle \mathbf{C}_n \mid n < \omega \rangle$ so that

(a) $\mathbf{C}_n \in V$ and it is a club in α in $V[\mathring{P}_\alpha]$.

(b) $\mathbf{C}_{n+1} \subseteq \mathbf{C}_n$.

(c) For every closed unbounded subset of α, $\mathbf{C} \in V[\mathring{P}_\alpha]$ there is some n so that $\mathbf{C}_n \subseteq \mathbf{C}$.

Let us define now a sequence $\langle q_n \mid n < \omega \rangle$, $q_n = \{\langle \tau, q_{n_\tau} \rangle \mid \tau \in i''_{\nu_n}(\beta_{q_n})\}$ so that

(i) $q_n \in Q_{\nu_n} \upharpoonright \alpha \cap V[\mathring{P}_{\beta_{q_n}+1}]$,

(ii) $\beta_{q_{n+1}} > \beta_{q_n} \geqq \beta_p$,

(iii) $\beta_{q_n} \in \mathbf{C}_n \cap A_{\nu_n} \cap C_\nu$,

(iv) $p_n \leqq_{\nu_n} q_n$,

(v) for every $\tau \in i''_{\nu_n}(\beta_{q_n})$, $\beta_{q_n} = \max q_{n_\tau}$,

(vi) q_{n+1} is stronger than some trivial extension of q_n,

(vii) q_n is a $\langle V[\mathring{P}_{\beta_{q_n}}], Q_{\nu_n} \upharpoonright \beta_{q_n} \rangle$-generic.

Using the inductive assumption and the fact that $A_{\nu_n} \cap \alpha$ is a stationary subset of α in V (since $A_{\nu_n} \supseteq \bar{A}_\nu$, $\alpha \in \bar{A}_\nu^{(1)}$ and so $\bar{A}_{\nu_\diamond} \cap \alpha = \{\beta < \alpha \mid \bar{A}_\nu \cap \beta = S_\beta$ and S_β is Wc_β-positive$\}$ is stationary, hence $\bar{A}_\nu \cap \alpha$ is stationary. Since $C \subseteq \alpha$ is a club, let us take some of its limit point $\beta \in \bar{A}_{\nu_\diamond} \cap \alpha$. Then C is a club in β and so $C \cap S_\beta = C \cap \bar{A}_\nu \cap \beta \neq \emptyset$), we can build such a sequence in $V[\mathring{P}_{\alpha+1}]$.

Let now $q = \{\langle \tau, q_\tau \rangle \mid \tau \in i''_\nu(\alpha)\}$ where $q_\tau = \bigcup \{q_{n\tau} \mid n < \omega$ and $\tau \in i''_{\nu_n}(\beta_{q_n})\} \cup \{\alpha\}$.

Let us prove that $q \in Q_\nu$, then clearly $q \geqq_\nu p$. Since $i''_\nu(\alpha) = \bigcup i''_{\nu_n}(\alpha)$ and every $q_n \in Q_{\nu_n}$, it is enough to show that for every n, $q \upharpoonright \langle \nu_n, \alpha \rangle = q \upharpoonright \nu_n$ is a $\langle V[P_\alpha], Q_{\nu_n} \upharpoonright \alpha \rangle$-generic.

The proof is similar to Proposition 3.1. Let $D \in V[\mathring{P}_\alpha]$ be a dense subset of $Q_{\nu_n} \upharpoonright \alpha$. Let us define in V an elementary chain $\langle \mathcal{M}_\beta \mid \beta < \alpha \rangle$ of submodels of $\langle V_{\kappa^{+++}}, \in, \alpha, \nu \rangle$ so that

(i) P_α, $\langle A_\mu \mid \mu < \kappa^+ \rangle$, i_ν, R_0, R_1 from the model $\mathcal{A}_{\alpha,\nu_n}$, $\langle \nu_{n_\tau} \upharpoonright \tau < \mathrm{cf}\, \nu_n \rangle$, the names $\mathbf{Q}_{\nu_n} \upharpoonright \alpha$ and $\mathbf{D}$ of $Q_{\nu_n} \upharpoonright \alpha$ and D are in $\mathcal{M}_0$.

(ii) Every $\mathcal{M}_\beta$ is of cardinality less than α.

(iii) $\mathcal{M}_{\beta+1}$ contains all β-sequences of elements of $\mathcal{M}_\beta$.

(iv) For limit β

$$\mathcal{M}_\beta = \bigcup_{\gamma<\beta} \mathcal{M}_\gamma.$$

Since α is an inaccessible cardinal and $V \models \mathrm{GCH}$, such a sequence can be defined.

Let $E = \{\beta < \alpha \mid \mathcal{M}_\beta \cap \alpha = \beta$ and $\beta \in C_\nu\}$. Then E is a club in V. Hence for some $m > n$, $\mathbf{C}_m \subseteq E$.

Then $\nu_n \in i''_{\nu_m}(\beta_{q_m})$, ν_n is a limit ordinal and $\beta_{q_m} \in q_{m\nu_n}$ so $q_m \upharpoonright \langle \nu_n, \beta_{q_m} \rangle = q_m \upharpoonright \nu_n$ is a $\langle V[\mathring{P}_{\beta_{q_m}}], Q_{\nu_n} \upharpoonright \beta_{q_m} \rangle$-generic. Let us prove that $q_m \upharpoonright \nu_n$ is stronger than some condition in D. It is enough to show that $D \cap Q_\nu \upharpoonright \beta_{q_m}$ belongs to $V[\mathring{P}_{\beta_{q_m}}]$ and it is dense in $Q_{\nu_n} \upharpoonright \beta_{q_m}$.

For the simplification let us drop the indexes n and q_m and denote ν_n by ν and β_{q_m} by β.

LEMMA 5.3. *For an inaccessible* $\beta \in E$

(1) $\mathscr{M}_\beta[\mathring{P}_\beta] < \langle V_{\kappa^{+++}}[\mathring{P}_\alpha], \in, \alpha, \nu\rangle$,

(2) $Q_\nu \restriction \alpha \cap \mathscr{M}_\beta[\mathring{P}_\beta] = Q_\nu \restriction \beta$, $D \cap \mathscr{M}_\beta[\mathring{P}_\beta] \in V[\mathring{P}_\beta]$ *and it is a dense subset of* $Q_\nu \restriction \beta$.

PROOF. (1) First note that $\mathscr{M}_\beta \cap P_\alpha = P_\beta$ and $\mathscr{M}_\beta \supseteq P_\beta$ since $\mathscr{M}_\beta \cap \alpha = \beta$, β is an inaccessible and for such β's, $P_\beta = \bigcup_{\gamma<\beta} P_\gamma$. If $S \subseteq P_\alpha$, $S \in \mathscr{M}_\beta$ and it is a maximal antichain in P_α iff $S \subseteq \mathscr{M}_\beta$ and $\mathscr{M}_\beta \models S$ is a maximal antichain in P_α, since P_α satisfies α-c.c. Hence $\mathring{P}_\beta$ is an $\mathscr{M}_\beta$-generic subset of P_α. Let $\mathscr{M}_\beta[\mathring{P}_\beta]$ be the $\mathring{P}_\beta$-interpretation of all the names which are in $\mathscr{M}_\beta$, i.e., it is $\{K_{\mathring{P}_\beta}(a) \mid a \in \mathscr{M}_\beta\}$. We can define $\Vdash$ inside $\mathscr{M}_\beta$. It will be the same as the forcing P_β in V restricted to the formulas whose quantifiers are bounded by $\mathscr{M}_\beta$. So $\mathscr{M}_\beta[\mathring{P}_\beta] \models \varphi(K_{\mathring{P}_\beta}(a))$ iff for some $p \in \mathring{P}_\beta$ in $\mathscr{M}_\beta$, $p \Vdash \varphi(a)$ iff $p \Vdash_{P_\alpha} \varphi(a)$ iff $V_{\kappa^{+++}}[\mathring{P}_\alpha] \models \varphi(K_{\mathring{P}_\alpha}(a))$. But a is really a P_β-name, so $K_{\mathring{P}_\alpha}(a) = K_{\mathring{P}_\beta}(a)$. Hence $\mathscr{M}_\beta[\mathring{P}_\beta] < \langle V_{\kappa^{+++}}[\mathring{P}_\alpha], \in, \alpha, \nu\rangle$.

(2) First, $Q_\nu \restriction \alpha \cap \mathscr{M}_\beta[\mathring{P}_\beta] \subseteq Q_\nu \restriction \beta$ follows from the definition of $Q_\nu \restriction \beta$ and since $\mathscr{M}_\beta[\mathring{P}_\beta] \subseteq V[\mathring{P}_\beta]$.

For the converse inclusion, note that if $t \subseteq \beta$, $t \in V[\mathring{P}_\beta]$ and it is bounded in β, then $t \in \mathscr{M}_\beta[\mathring{P}_\beta]$. Since for some $\xi < \beta$, $\bigcup t = \xi$, then ξ is of cardinality $\aleph_1$ in $V[\mathring{P}_\beta]$ and P_β satisfies β-c.c. so $t \in V[\mathring{P}_\eta]$ for some $\eta < \beta$. Hence some of its names can be coded as an ordinal less than $(\eta^+)^V$. But $(\eta^+)^V < \beta$. Hence this name belongs to $\mathscr{M}_\beta$ and so $t \in \mathscr{M}_\beta$.

The second half of (2) follows now from the first and (1).

☐ of Lemma 5.3.

☐ of Case 2.

Case 3. $\mathrm{cf}^{V[\mathring{P}_\kappa]} \nu = \aleph_1$.

As in Case 2 we have $\alpha \in C_\nu$ so $\alpha > \mathrm{cf}^V \nu$,

$$i''_\nu(\alpha) = \bigcup_{\tau < \mathrm{cf}^V \nu < \alpha} i''_{\nu_\tau}(\alpha),$$

$i''_{\nu_{\tau+1}}(\alpha) \supseteq i''_{\nu_\tau}(\alpha)$ and $\nu_\tau \in i''_{\nu_{\tau+1}}(\alpha)$ for $\tau < \mathrm{cf}^V \alpha$.

Let us pick in $V[\mathring{P}_\alpha]$ a cofinal in ν continuous sequence of limit ordinals $\langle \nu_i \mid i < \omega_1\rangle$. Let it be a subsequence of $\langle \nu_\tau \mid \tau < \mathrm{cf}^V \nu\rangle$. Then $\{\nu_i \mid i < \omega_1\} \subseteq i''_\nu(\alpha)$, $i''_\nu(\alpha) = \bigcup_{i<\omega_1} i''_{\nu_i}(\alpha)$ and $\nu_i \in i''_{\nu_{i+1}}(\alpha)$. Note that the same is true for any other ordinal in C_ν.

Let $p = \{\langle \tau, p_\tau\rangle \mid \tau \in i''_\nu(\beta_p)\} \in Q_\nu \restriction \alpha$ where $\beta_p < \alpha$ and $\beta_p \in C_\nu$. W.l.o.g. $p \in V[\mathring{P}_{\beta_p}]$. Since $p \in V[\mathring{P}_\alpha]$, P_α satisfies α-c.c. and $|p|^{V[\mathring{P}_\alpha]} = \aleph_1$. So $p \in V[\mathring{P}_\beta]$ for some $\beta < \alpha$. Now let us take the trivial extension p' of p with $\beta_{p'} = \beta$. Then p' satisfies this requirement.

Let us denote $p \upharpoonright \nu_i$ by p_i for $i < \omega_1$. Then $p_i \in Q_{\nu_i} \upharpoonright \alpha$ and $p = \bigcup_{i<\omega_1} p_i$.

Let $\langle \mathbf{C}_n \mid n < \omega \rangle \in V[\mathring{P}_{\alpha+1}]$ be the sequence of clubs defined in the beginning of Case 2.

$\alpha \in \bar{A}_\nu^{(1)}$ so $(\bar{A}_\nu)_\diamond$ is stationary in α. Pick $\tau_0 \in (\bar{A}_\nu)_\diamond \cap (\mathbf{C}_0 - \beta_p) \cap C_\nu$ such that $\tau_0 \cap \mathbf{C}_0$ is unbounded and so closed unbounded in τ_0.

It follows from the definition of P_κ that $P_{\tau_0+1} = P_{\tau_0} * \underset{\sim}{Q}_{\tau_0}$, where $\underset{\sim}{Q}_{\tau_0}$ is $P^*[\bar{A}_\nu \cap \tau_0]$ (see Section 2).

So $\bigcup \underset{\sim}{\mathring{Q}}_{\tau_0} \cap C_\nu \cap (\mathbf{C}_0 - \beta_p)$ is a club in τ_0 in $V[\mathring{P}_{\tau_0+1}]$. We shall denote it by G_0.

Let $\{\alpha_i \mid i < \aleph_1\}$ be the increasing continuous enumeration of G_0 in $V[\mathring{P}_{\tau_0+1}]$. Note that for every limit $i < \aleph_1$, $G_0 \cap \alpha_i \in V[\mathring{P}_{\alpha_i+1}]$. Since it is a countable subset of α_i and the cardinality of α_i is $\aleph_1$, in $V[\mathring{P}_{\alpha_i+1}]$. The forcing $P_\kappa / \mathring{P}_{\alpha_i+1}$ does not add new ω-sequences to ordinals of cardinality $\aleph_1$, since P_κ does not add reals.

Since $\alpha_0 \in A_{\nu_0} \cap C_{\nu_0}$ we can apply the inductive assumption to p_0 (which belongs to $Q_{\nu_0} \upharpoonright \alpha_0$) and α_0. So there is $t_0 \in V[\mathring{P}_{\alpha_0+1}] \cap Q_{\nu_0}$, $t_0 \geqq_{\nu_0} p_0$, $\max t_{0\tau} = \alpha_0$ for every $\tau \in i''_{\nu_0}(\alpha_0)$ and t_0 is a $\langle V[\mathring{P}_{\alpha_0}], Q_{\nu_0} \upharpoonright \alpha_0 \rangle$-generic. Let $t'_0 = t_0 \cup p_1$. Then $t'_0 \in Q_{\nu_1} \upharpoonright \alpha_1$. There is $t_1 \in V[\mathring{P}_{\alpha_1+1}] \cap Q_{\nu_1}$, $t_1 \geqq_{\nu_1} t'_0$, $\max t_{1\tau} = \alpha_1$, for $\tau \in i''_{\nu_1}(\alpha_1)$, and t_1 is a $\langle V[\mathring{P}_{\alpha_1}], Q_{\nu_1} \upharpoonright \alpha_1 \rangle$-generic. Let $t'_1 = t_1 \cup p_2$.

In such a way we obtain a sequence $\langle t_\gamma \mid \gamma < \aleph_1 \rangle$ so that for any $\gamma < \aleph_1$

(1) $t_\gamma \in V[\mathring{P}_{\alpha_\gamma+1}] \cap Q_{\nu_\gamma}$, $t_\gamma = \{\langle \tau, t_{\gamma\tau} \rangle \mid \tau \in i''_{\nu_\gamma}(\alpha_\gamma)\}$.

(2) $\max t_{\gamma\tau} = \alpha_\gamma$ for $\tau \in i''_{\nu_\gamma}(\alpha_\gamma)$.

(3) $t_\gamma \geqq_{\nu_\gamma} p_\gamma$.

(4) $t_{\gamma+1} \geqq_{\nu_{\gamma+1}} t_\gamma \cup p_{\gamma+1}$.

As we saw, there is no problem to build such a sequence on nonlimit stages.

Suppose that $\{t_{\gamma'} \mid \gamma' < \gamma \rangle$ is built and γ is a limit ordinal less than $\aleph_1$. Let $t_\gamma = \{\langle \tau, t_{\gamma\tau} \rangle \mid \tau \in i''_{\nu_\gamma}(\alpha_\gamma)\}$ where $t_{\gamma\tau} = \bigcup \{t_{\gamma'\tau} \mid \tau \in i''_{\nu_{\gamma'}}(\alpha_{\gamma'})$ and $\gamma' < \gamma\} \cup \{\alpha_\gamma\}$. t_γ is in $V[\mathring{P}_{\alpha_\gamma+1}]$, since the sequence $\langle \alpha_{\gamma'} \mid \gamma' < \gamma \rangle$ is countable and so belongs to $V[\mathring{P}_{\alpha_\gamma+1}]$. $\max t_{\gamma\tau} = \bigcup \max_{\gamma'<\gamma} t_{\gamma'\tau} = \alpha_\gamma$. for every $\tau \in i''_{\nu_\gamma}(\alpha_\gamma)$.

Claim. $t_\gamma \in Q_{\nu_\gamma}$.

Proof. We shall check condition (2) from the main definition. So suppose that limit $\tau \in i''_{\nu_\gamma}(\alpha_\gamma)$ and $\beta \in t_{\gamma\tau}$. Since $i''_{\nu_\gamma}(\alpha_\gamma) = \bigcup_{\gamma'<\gamma} i''_{\nu_{\gamma'}}(\alpha_\gamma)$, $\tau \in i''_{\nu_{\gamma'}}(\alpha_\gamma)$ for some $\gamma' < \gamma$.

Subcase 1. $\beta < \alpha_\gamma$.

Then for some $\delta < \gamma$, $\nu_\delta > \nu_{\gamma'}$ and $\alpha_\delta > \beta$. So $i''_{\nu_\delta}(\alpha_\delta) \supseteq i''_{\nu_{\gamma'}}(\beta)$. Also $\tau \in i''_{\nu_\delta}(\alpha_\delta)$ and $\beta \in t_{\delta\tau}$, since $\max t_{\delta\tau} = \alpha_\delta > \beta$. But $t_\delta \in Q_{\nu_\delta}$, so $t_\delta \upharpoonright \langle \tau, \beta \rangle$ is a $\langle V[\mathring{P}_\beta], Q_\tau \upharpoonright \beta \rangle$-generic and $t_\delta \upharpoonright \langle \tau, \beta \rangle = t_\gamma \upharpoonright \langle \tau, \beta \rangle$.

Subcase 2. $\beta = \alpha_\gamma$.

We shall prove that $t_\gamma \restriction \langle \tau, \alpha_\gamma \rangle$ is a $\langle V[\mathring{P}_{\alpha_\gamma}], Q_\tau \restriction \alpha_\gamma \rangle$-generic. So suppose $D \in V[\mathring{P}_{\alpha_\gamma}]$ is a dense subset of $Q_\tau \restriction \alpha_\gamma$. As in Case 2 we define an elementary chain $\{\mathcal{M}_\delta \mid \delta < \alpha_\gamma \rangle$ of submodels of $\langle V_{\kappa^{+++}}, \in, \alpha_\gamma, \tau \rangle$, which satisfies (i)–(iv) and $E = \{\delta < \alpha_\gamma \mid \mathcal{M}_\delta \cap \alpha_\gamma = \delta \text{ and } \delta \in C_\nu\}$.

Now $\alpha_\gamma \in G_0$ and it is its limit point. Hence it is also a limit point of $\mathbf{C}_0 - \beta_p$. So $\mathbf{C}_0 \cap \alpha_\gamma$ is a closed unbounded subset. Then $E \cap (\mathbf{C}_0 - \beta_p)$ is also a club in α_γ. By the definition of $P^*[\bar{A}_\nu \cap \tau_0]$, then $\bigcup \breve{\underset{\sim}{Q}}_{\tau_0} \cap \alpha_\gamma$ intersects with every closed unbounded subset of α_γ in $V[\breve{P}_{\alpha_\gamma}]$, so $\bigcup \breve{\underset{\sim}{Q}}_{\tau_0} \cap E \cap (\mathbf{C}_0 - \beta_p) \neq \emptyset$. Hence for some $\mu < \gamma$, α_μ belongs to this intersection and $\tau \in i''_{\nu_\mu}(\alpha_\mu)$.

Now by Lemma 5.3, $Q_\tau \restriction \alpha_\gamma \cap \mathcal{M}_{\alpha_\mu}[\breve{P}_{\alpha_\mu}] = Q_\tau \restriction \alpha_\mu$, $D \cap \mathcal{M}_{\alpha_\mu}[\breve{P}_{\alpha_\mu}] \in V[\breve{P}_{\alpha_\mu}]$ and it is a dense subset of $Q_\tau \restriction \alpha_\mu$. Then $t_\mu \restriction \langle \tau, \alpha_\mu \rangle$ and so $t_\gamma \restriction \langle \tau, \alpha_\gamma \rangle$ contains some element of D. ☐ of Subcase 2.

Let now $t^0 = \{\langle \tau, t^0_\tau \rangle \mid \tau \in \bigcup_{\gamma < \aleph_1} i''_{\nu_\gamma}(\alpha_\gamma) = i''_\nu(\tau_0)\}$ where $t^0_\tau = \bigcup \{t_{\gamma\tau} \mid \tau \in i''_{\nu_\gamma}(\alpha_\gamma)$ and $\gamma < \aleph_1\}$ and τ_0 is from the beginning of the proof of Case 3. Note that for every $\tau \in i''_\nu(\tau_0)$, $\bigcup t^0_\tau = \tau_0$.

Now let us pick some $\tau_1 \in (\bar{A}_\nu)_\diamond \cap (\mathbf{C}_1 - \tau_0) \cap C_\nu$, so that $\tau_1 \cap \mathbf{C}_1$ is unbounded in τ_1.

As above $P_{\tau_1 + 1} = P_{\tau_1} * \underset{\sim}{Q}_{\tau_1}$ where $\underset{\sim}{Q}_{\tau_1}$ is $P^*[\bar{A}_\nu \cap \tau_1]$. Let $G_1 = (\bigcup \underset{\sim}{Q}_{\tau_1}) \cap (\mathbf{C}_1 - \tau_0) \cap C_\nu$. Let $\{\alpha^1_i \mid i < \aleph_1\}$ be the increasing continuous enumeration of G_1 in $V[\breve{P}_{\tau_1 + 1}]$.

As above we build the sequence $\langle t^1_\gamma \mid \gamma < \aleph_1 \rangle$ so that for $\gamma < \aleph_1$

(1) $t^1_\gamma \in V[\breve{P}_{\alpha^1_\gamma + 1}] \cap Q_{\nu_\gamma} t^1_\gamma = \{\langle \tau, t^1_{\gamma\tau} \rangle \mid \tau \in i''_{\nu_\gamma}(\alpha^1_\gamma)\}$.

(2) $\max t^1_{\gamma\tau} = \alpha^1_\gamma$ for $\tau \in i''_{\nu_\gamma}(\alpha^1_\gamma)$.

(3) $t^1_{\gamma\nu_\gamma} \geqq t_\gamma$.

(4) $t^1_{\gamma+1\, \nu_{\gamma+1}} \geqq t^1_\gamma \cup t_{\gamma+1}$.

Let us define

$$t^1 = \left\{ \langle \tau, t^1_\tau \rangle \,\middle|\, \tau \in \bigcup_{\gamma < \aleph_1} i''_{\nu_\gamma}(\alpha^1_\gamma) = i''_\nu(\tau_1) \right\}$$

where $t^1_\tau = \bigcup \{t^1_{\gamma\tau} \mid \tau \in i''_{\nu_\gamma}(\alpha^1_\gamma)$ and $\gamma < \aleph_1\}$.

Then for any $\tau \in i''_\nu(\tau_1)$, $\bigcup t^1_\tau = \tau_1$.

In the same way let us define τ_n, t^n for every $n < \omega$. Let now $t^\omega = \{\langle \tau, t^\omega_\tau \rangle \mid \tau \in i''_\nu(\alpha)\}$, where $t^\omega_\tau = \bigcup \{t^n_{\gamma\tau} \mid \tau \in i''_{\nu_\gamma}(\alpha^n_\gamma), \gamma < \aleph_1 \text{ and } n < \omega\} \cup \{\alpha\}$. Note that $\bigcup_{\gamma < \aleph_1} i''_{\nu_\gamma}(\alpha) = i''_\nu(\alpha)$ and $\bigcup_{n < \omega} i''_{\nu_\gamma}(\alpha^n_\gamma) = i''_{\nu_\gamma}(\alpha)$.

It remains to show that t^ω is in Q_ν.

It is enough to prove the following:

CLAIM. *For every limit $\mu \in i''_\nu(\alpha)$, $t^\omega \restriction \langle \mu, \alpha \rangle$ is a $\langle Q_\mu \restriction \alpha, V[\mathring{P}_\alpha] \rangle$-generic.*

PROOF. The proof of this fact is similar to Subcase 2. Let $D \in V[\mathring{P}_\alpha]$ be a dense subset of $Q_\mu \upharpoonright \alpha$. Let $\langle \mathcal{M}_\gamma \mid \delta < \alpha \rangle$ and E be as in Subcase 2. Let $\mathbf{C}_n \subseteq E$. Then for every $m \geqq n$, $\mathbf{C}_m \subseteq \mathbf{C}_n$ and $\{\alpha_i^m \mid i < \aleph_1\} \subseteq \mathbf{C}_m \cap \tau_m$. Hence for every $i < \aleph_1$, by Lemma 5.3,

$$Q_\mu \upharpoonright \alpha \cap \mathcal{M}_{\alpha_i^m}[\mathring{P}_{\alpha_i^m}] = Q_\mu \upharpoonright \alpha_i^m,$$

$D \cap \mathcal{M}_{\alpha_i^m}[\mathring{P}_{\alpha_i^m}] \in V[\mathring{P}_{\alpha_i^m}]$ and it is a dense subset of $Q_\mu \upharpoonright \alpha_i^m$.

Our $\mu \in i''_\nu(\alpha)$. So for some $i < \aleph_1$ and $m \geqq n$, $\mu \in i''_{\nu_i}(\alpha_i^m)$. But $t_i^m \in Q_{\nu_i}$ and $\alpha_i^m \in t_{i\mu}^m$. Hence $t_i^m \upharpoonright \langle \mu, \alpha_i^m \rangle$ is a $\langle V[\mathring{P}_{\alpha_i^m}], Q_\mu \upharpoonright \alpha_i^m \rangle$-generic. So $t_i^m \upharpoonright \langle \mu, \alpha_i^m \rangle$ is stronger than some element of D. Hence $t^\omega \upharpoonright \langle \mu, \alpha \rangle$ satisfies the same.

☐ of Case 3.

Case 4. $\mathrm{cf}^V \nu = \kappa$.

PROOF. Let $\langle \nu_\mu \mid \mu < \kappa \rangle$ be the picked cofinal sequence for ν. Then $i''_\nu(\beta) = \bigcup_{\mu < \beta} i''_{\nu_\mu}(\beta)$ and for $\mu < \beta$, $\nu_\mu \in i''_\nu(\beta)$ for every $\beta \in C_\nu$. Hence by Lemma 4.2, $\beta \in C_\mu$ for every $\mu < \beta$.

Let $p = \{\langle \tau, p_\tau \rangle \mid \tau \in i''_\nu(\beta_p)\} \in Q_\nu \upharpoonright \alpha$ where $\beta_p < \alpha$ and $\beta_p \in C_\nu$. Our α is regular, so for some $\mu < \alpha$, $i''_{\nu_\mu}(\alpha) \supseteq i''_\nu(\beta_p)$. W.l.o.g. let already $i''_{\nu_0}(\alpha) \supseteq i''_\nu(\beta_p)$.

Let $\langle \mathbf{C}_n \mid n < \omega \rangle \in V[\mathring{P}_{\alpha+1}]$ be as above.

Using the inductive assumption, as in Case 2, we define a sequence $\langle q_n \mid n < \omega \rangle$, $q_n = \{\langle \tau, q_{n\tau} \rangle \mid \tau \in i''_{\nu_{\bar{\mu}_n}}(\mu_n)\}$ so that

(i) $q_n \in Q_{\nu_{\bar{\mu}_n}} \upharpoonright \alpha \cap V[\mathring{P}_{\mu_{n+1}}]$.

(ii) $\bar{\mu}_0 = 0$ and $\bar{\mu}_n < \mu_n < \bar{\mu}_{n+1} < \alpha$.

(iii) $\mu_n \in \mathbf{C}_n \cap A_{\nu_{\bar{\mu}_n}} \cap C_\nu$.

(iv) $q_{0\,\nu_0} \geqq p$.

(v) For every $\tau \in i''_{\nu_{\bar{\mu}_n}}(\mu_n)$, $\mu_n = \max q_{n\tau}$.

(vi) q_{n+1} is stronger than some trivial extension of q_n.

(vii) q_n is a $\langle V[\mathring{P}_{\mu_n}], Q_{\nu_{\bar{\mu}_n}} \upharpoonright \mu_n \rangle$-generic.

Note that $\bar{A}_\nu = \triangle_{\mu \leq \kappa} A_{\nu_\mu}$ and since $\alpha \in \bar{A}_\nu^{(1)}$, $\bar{A}_\nu \cap \alpha$ is stationary. So for $\mu < \alpha$, $A_{\nu_\mu} \cap \alpha \supseteq (\bar{A}_\nu - \mu) \cap \alpha$ is stationary. Also $\mu_n \in C_{\nu_{\bar{\mu}_n}}$ for every n since $\mu_n > \bar{\mu}_n$ and $\mu_n \in C_\nu$.

Let now $q = \{\langle \tau, q_\tau \rangle \mid \tau \in i''_\nu(\alpha)\}$ where $q_\tau = \bigcup \{q_{n\tau} \mid n < \omega$ and $\tau \in i''_{\nu_{\bar{\mu}_n}}(\mu_n)\} \cup \{\alpha\}$.

As above such defined q belongs to $V[\mathring{P}_{\alpha+1}]$. Let us show that q belongs to Q_ν. So suppose $\delta \in i''_\nu(\alpha)$ is a limit ordinal and $\beta \in q_\delta$. We shall prove that $q \upharpoonright \langle \delta, \beta \rangle$ is a $\langle V[\mathring{P}_\beta], Q_\delta \upharpoonright \beta \rangle$-generic. First note that if $\beta < \alpha$, then $i''_\delta(\beta) \subseteq i''_{\nu_{\bar{\mu}_n}}(\mu_n)$ and $\delta \in i''_{\nu_{\bar{\mu}_n}}(\mu_n)$, for some $n < \omega$. So $q \upharpoonright \langle \delta, \beta \rangle = q_n \upharpoonright \langle \delta, \beta \rangle$. But $q_n \in Q_{\nu_{\bar{\mu}_n}}$, $\delta \in i''_{\nu_{\bar{\mu}_n}}(\mu_n)$ is a limit ordinal and $\beta \in q_{n\delta}$, since $\max q_{n\delta} = \mu_n > \beta$, hence $q_n \upharpoonright \langle \delta, \beta \rangle$ is $\langle V[\mathring{P}_\beta], q_\delta \upharpoonright \beta \rangle$-generic.

There remains the case $\beta = \alpha$. The proof is the same as in Claim of Case 3.

$\square$ of Case 4.

$\square$ of the lemma.

Now let us return to the proposition.

So we have $\alpha \in A_\nu \cap C_\nu$ and $p \in Q_\nu \restriction \alpha$. Let us assume that $p = \emptyset$. In the general case only the notations are more complicated.

In $V[\mathring{P}_{\alpha+1}]$, $\operatorname{cf} \alpha = \operatorname{cf}(\alpha^+) = \aleph_0$. So there is a sequence $\langle B_n \mid n < \omega \rangle$ so that:

(1) every B_n belongs to $V[\mathring{P}_\alpha]$ and it is a one-to-one function from $\aleph_1$ into the set of dense subsets of $Q_\nu \restriction \alpha$. (Note that $\aleph_1^{V[\mathring{P}_\kappa]} = \aleph_1^V$.)

(2) For every dense subset $D \in V[\mathring{P}_\alpha]$ of $Q_\nu \restriction \alpha$ there are $n < \omega$ and $\tau < \aleph_1$ so that $B_n(\tau) = D$.

As we did in the lemma, let us define in V an elementary chain $\langle \mathcal{M}_\beta \mid \beta < \alpha \rangle$ of submodels of $\langle V_{\kappa^{+++}}, \in, \alpha, \nu \rangle$ so that

(i) P_α, $\langle A_\mu \mid \mu < \kappa^+ \rangle$, i_ν, R_0, R_1 from the model $\mathcal{A}_{\alpha,\nu}$, $\langle \nu_\tau \mid \tau < \operatorname{cf} \nu \rangle$, the names $\underset{\sim}{Q_\nu \restriction \alpha}$, $\underset{\sim}{B_0}$ of $Q_\nu \restriction \alpha$ and B_0 are in $\mathcal{M}_0$.

(ii) Every $\mathcal{M}_\beta$ is of cardinality less than α.

(iii) $\mathcal{M}_{\beta+1}$ contains all β-sequences of elements of $\mathcal{M}_\beta$.

(iv) For limit β, $\mathcal{M}_\beta = \bigcup_{\gamma<\beta} \mathcal{M}_\gamma$.

Let $E_0 = \{\beta < \alpha \mid \mathcal{M}_\beta \cap \alpha = \beta \text{ and } \beta \in C_\nu\}$. Let us pick some limit point γ_0 of E_0, which belongs to $(\bar{A}_\nu^{(1)})_\diamond$. There is such an ordinal, since $\alpha \in \bar{A}_\nu^{(2)} = A_\nu$ and so $(\bar{A}_\nu^{(1)})_\diamond \cap \alpha$ is stationary in α.

On the step γ_0 we forced an ω-club $\bigcup \underset{\sim}{\mathring{Q}}_{\gamma_0}$ into $\bar{A}_\nu^{(1)} \cap \gamma_0$. Let $G_0 = E_0 \cap (\bigcup \underset{\sim}{\mathring{Q}}_{\gamma_0})$. So G_0 is a club in γ_0 in $V[\mathring{P}_{\gamma_0+1}]$ and γ_0 became an ordinal of cofinality $\aleph_1$ in this world. Let $\{\alpha_i \mid i < \aleph_1\}$ be the increasing continuous enumeration of G_0 in $V[\mathring{P}_{\gamma_0+1}]$. As we explained in Case 2 of the lemma, for every limit $i < \aleph_1$, $G_0 \cap \alpha_i \in V[\mathring{P}_{\alpha_i+1}]$.

$\mathcal{M}_{\gamma_0}[\mathring{P}_{\gamma_0}] = \bigcup \{\mathcal{M}_{\alpha_i}[\mathring{P}_{\alpha_i}] \mid i < \aleph_1\}$. As in Lemma 5.3 for every inaccessible $\beta \in E_0$, $B_{0\beta} =_{\mathrm{df}} B_0 \cap \mathcal{M}_\beta[\mathring{P}_\beta] \in V[\mathring{P}_\beta]$ and for every $i < \aleph_1$, $B_{0\beta}(i)$ is a dense subset of $Q_\nu \restriction \beta$.

So $B_{0\gamma_0}(\xi) = \bigcup \{B_{0\alpha_i}(\xi) \mid i < \aleph_1\}$ for every $\xi < \aleph_1$.

Now let us define in $V[\mathring{P}_{\gamma_0+1}]$ a sequence $\langle q_i \mid i < \aleph_1 \rangle$, so that for every $i < \aleph_1$

(i) $q_i = \{\langle \tau, q_{i\tau} \rangle \mid \tau \in i_\nu''(\alpha_i)\}$.

(ii) $\max q_{i\tau} = \alpha_i$ for every $\tau \in i_\nu''(\alpha_i)$.

(iii) $q_i \in Q_\nu \cap V[\mathring{P}_{\alpha_i+1}]$.

(iv) q_{i+1} is stronger than some element of $B_{0\gamma_0}(i)$.

(v) $q_{i+1} {}_\nu{\geqq} q_i$.

Let q_0 be any element that satisfies (i)–(iii). It exists by the lemma. Note that $q_0 \in \mathcal{M}_{\alpha_1}[\mathring{P}_{\alpha_1}]$, since $q_0 \in Q_\nu \restriction \alpha_1$ which is by Lemma 5.3 $Q_\nu \restriction \alpha \cap \mathcal{M}_{\alpha_1}[\mathring{P}_{\alpha_1}]$. Now let

p be any element of $B_{0\alpha_1}(0)$ stronger than q_0 (clearly, it exists since $B_{0\alpha_1}(0)$ is dense in $Q_\nu \restriction \alpha \cap \mathcal{M}_{\alpha_1}[\mathring{P}_{\alpha_1}]$). By Lemma 5.2 there is q which satisfies (i)–(iii) and $q \geqq_\nu p$. Let q_1 be some such q.

So for every non-limit i it is possible to define q_i in such a way.

Now suppose i is a limit ordinal less than $\aleph_1$. Let us define $q_i = \{\langle \tau, q_{i\tau}\rangle \mid \tau \in i''_\nu(\alpha_i)\}$, where $q_{i\tau} = \bigcup \{q_{\xi\tau} \mid \xi < i$ and $\tau \in i''_\nu(\alpha_\xi)\} \cup \{\alpha_i\}$ for $\tau \in i''_\nu(\alpha_i)$. Let us check that (iii) holds, i.e., $q_i \in Q_\nu \cap V[\mathring{P}_{\alpha_i+1}]$. First note that q_i is in $V[\mathring{P}_{\alpha_i+1}]$ since we used only $\{\alpha_\xi \mid \xi < i\}$ to build it. And it is a countable sequence of ordinals less than α_i. So it belongs to $V[\mathring{P}_{\alpha_i+1}]$. The proof that $q_i \in Q_\nu$ is the same as in Lemma 5.2, Cases 2 and 3.

Now let $q^0 = \{\langle \tau, q^0_\tau\rangle \mid \tau \in \bigcup_{i<\aleph_1} i''_\nu(\alpha_i)\}$, where $q^0_\tau = \bigcup \{q_{i\tau} \mid i < \omega_1$ and $\tau \in i''_\nu(\alpha_i)\}$. Note that $\bigcup_{i<\aleph_1} i''_\nu(\alpha_i) = i''_\nu(\gamma_0)$. The argument similar to those in Lemma 5.2 shows that $q^0 \in Q_\nu$. Also note that q^0 is built inside $V[\mathring{P}_{\gamma_0+1}]$, so it belongs to $V[\mathring{P}_{\gamma_0+1}]$. Clearly, then $q^0 \in Q_\nu \restriction \alpha$.

Let us consider now an elementary chain $\langle \mathcal{M}^1_\beta \mid \beta < \alpha\rangle$ of submodels of $\langle V_{\kappa^{+++}}, \in, \alpha, \nu\rangle$ which satisfies (i)–(iv) as above and, in addition, in (i) we include also some name $\underline{B}_1$ of B_1 into $\mathcal{M}^1_0$.

As before, let $E_1 = \{\beta < \alpha \mid \mathcal{M}_\beta \cap \alpha = \beta$ and $\beta \in C_\nu\}$. Pick some limit point γ_1 of E, so that $\gamma_1 \in (\bar{A}^{(1)}_\nu)_\diamond$ and $\gamma_1 > \gamma_0$. Let $G_1 = E_1 \cap (\bigcup \underset{\sim}{\mathring{Q}}_{\gamma_1})$ and $\{\alpha^1_i \mid i < \aleph_1\}$ be its increasing continuous enumeration in $V[\mathring{P}_{\gamma_1+1}]$. Then

$$\mathcal{M}^1_{\gamma_1} = \bigcup \{\mathcal{M}^1_{\alpha^1_i} \mid i < \aleph_1\}$$

and

$$\mathcal{M}^1_{\gamma_1}[\mathring{P}_{\gamma_1}] = \bigcup \{\mathcal{M}^1_{\alpha^1_i}[\mathring{P}_{\alpha^1_i}] \mid i < \aleph_1\}.$$

Now we define $\langle q^1_i \mid i < \aleph_1\rangle$ satisfying (i)–(iii) and (v) as above. We only change α_i on α^1_i and (iv) will be the following: q^1_{i+1} is stronger than some element of $B_{1\gamma_1}(i)$. Also let us pick q^1_0 to be stronger than q^0. Now, as before, we define q^1. Such q^1 belongs to $Q_\nu \restriction \alpha \cap V[\mathring{P}_{\gamma_1+1}]$.

Let us do this construction for every $n < \omega$. So we obtain the sequence $\langle q^n \mid n < \omega\rangle$. Let $q = \{\langle \tau, q_\tau\rangle \mid \tau \in i''_\nu(\alpha)\}$ where $q_\tau = \bigcup \{q^n_\tau \mid n < \omega$ and $\tau \in i''_\nu(\gamma_n)\} \cup \{\alpha\}$. Such defined $q \in V[\mathring{P}_{\alpha+1}]$. By its definition q is stronger than some element of every $D \in V[\mathring{P}_\alpha]$, where D is a dense subset of $Q_\nu \restriction \alpha$. So $q \in Q_\nu$ and it is $\langle V[\mathring{P}_\alpha], Q_\nu \restriction \alpha\rangle$-generic. ($q \in Q_\nu$ since for every limit $\mu \in i''_\nu(\alpha)$, as in Lemma 4.5, $Q_\mu \restriction \alpha \lessdot Q_\nu \restriction \alpha$. So $q \restriction \langle \mu, \alpha\rangle$ is $\langle V[\mathring{P}_\alpha], Q_\mu \restriction \alpha\rangle$-generic.)

For $\bigcup (q_\tau \cap \alpha) = \alpha$, note that $\bigcup_{n<\omega} \gamma_n = \alpha$, since for every $\beta < \alpha$, $D_\beta = \{p \in Q_\nu \restriction \alpha \mid \exists \tau \in i''_\nu(\beta_p) \beta \leqq \bigcup p_\tau\}$ is a dense subset of $Q_\nu \restriction \alpha$. (We can add this β or some ordinal $\geqq \beta$ to p_τ for nonlimit τ.) $\square$ of Proposition 5.1.

PROPOSITION 5.4. *For every limit* $\nu < \kappa^+$ *and an ordinal* $\alpha \in A_\nu \cap C_\nu$ *the forcing* $Q_\nu \restriction \alpha$ *over* $V[\mathring{P}_\alpha]$ *does not add new functions on* $\aleph_1$ *into* $V[\mathring{P}_\alpha]$.

PROOF Suppose $\underset{\sim}{f} \in V[\mathring{P}_\alpha]$ is a name of such a function. Let us define $B_0(i) = \{q \in Q_\nu \restriction \alpha \mid \exists a \in V[\mathring{P}_\alpha] q \Vdash_{Q_\nu \restriction \alpha} \underset{\sim}{f}(i) = a\}$ for $i < \aleph_1$. Then for every $i < \aleph_1$, $B_0(i)$ is a dense subset of $Q_\nu \restriction \alpha$. As in Proposition 5.1, let us build $q^0 \in Q_\nu \restriction \alpha$. But then already q^0 knows every value of f, i.e. $q^0 \Vdash_{Q_\nu \restriction \alpha} \underset{\sim}{f} \in V[\mathring{P}_\alpha]$. $\square$

REMARK. We need the assumption $\alpha \in A_\nu \cap C_\nu$ for a limit ν, since otherwise, for some $\mu \in i''_\nu(\alpha)$, $A_\mu \cap \alpha$ may be nonstationary and then $Q_\nu \restriction \alpha$ collapses $\aleph_2$.

Let $N = V^\kappa/\mathcal{U}$ and $j : V \to N$ be the elementary embedding.

PROPOSITION 5.5. *For every limit* $\nu < \kappa^+$, *in* $N[\mathring{P}_{\kappa+1}]$ *there is a* $\langle V[\mathring{P}_\kappa], Q_{j(\nu)} \restriction \kappa\rangle$*-generic set* q *so that* $q \in Q_{j(\nu)}$, $q = \{\langle \tau, q_\tau\rangle \mid \tau \in j''(\nu)\}$ *and* $\bigcup (q_\tau \cap \kappa) = \kappa$ *for every* $\tau \in j''(\nu)$.

PROOF. This proposition is the translation of Proposition 5.1 to N. Note only that $j''(\nu) = i''_{j(\nu)}(\kappa)$, $\kappa \in j(A_\nu \cap C_\nu) = A_{j(\nu)} \cap C_{j(\nu)}$ (since $A_\nu \cap C_\nu \in \mathcal{U}$ and $\mathcal{U}$ is normal) and the $\langle V[\mathring{P}_\kappa]; Q_{j(\nu)} \restriction \kappa\rangle$ and the $\langle N[\mathring{P}_\kappa], Q_{j(\nu)} \restriction \kappa\rangle$ genericity are the same, since N is closed under κ-sequences of its elements. $\square$

LEMMA 5.6. *For limit* $\nu < \kappa^+$ *and* $\alpha \in A_\nu \cap C_\nu$ *or* $\alpha = \kappa$, $Q_\nu \restriction \alpha$ *is isomorphic to* $Q_{j(\nu)} \restriction \alpha$.

REMARK. (1) Since $V[\mathring{P}_\kappa] \cap {}^\kappa N[\mathring{P}_\kappa] \subseteq N[\mathring{P}_\kappa]$ this isomorphism is in $N[P_\kappa]$.
(2) $Q_\nu \restriction \kappa = Q_\nu$.

PROOF. Let $q \in Q_\nu \restriction \alpha$, $q = \{\langle \tau, q_\tau\rangle \mid \tau \in i''_\nu(\beta_q)\}$ where $\beta_q < \alpha \leqq \kappa$. Let us define $\varphi(q)$ to be $\{\langle j(\tau), q_\tau\rangle \mid \tau \in i''_\nu(\beta_q)\}$. Then $\varphi(q) = \{\langle \tau, \bar{q}_\tau\rangle \mid \tau \in i''_{j(\nu)}(\beta_q)\}$ where $\bar{q}_{j(\tau)} = q_\tau$. Note that $i''_{j(\nu)}(\beta_q) = j(i''_\nu(\beta_q)) = \{j(\tau) \mid \tau \in i''_\nu(\beta_q)\}$ since $\beta_q < \kappa$. Since $N[\mathring{P}_\alpha] \supseteq V[\mathring{P}_\alpha] \cap {}^\kappa N[\mathring{P}_\alpha]$, $\varphi(q) \in N[\mathring{P}_\alpha]$. By induction on ν it is easy to check that $\varphi(q) \in Q_{j(\nu)} \restriction \alpha$. $\square$

PROPOSITION 5.7. *For every limit* $\nu < \kappa^+$ *in* $N[\mathring{P}_{\kappa+1}]$ *there is a* $\langle V[\mathring{P}_\kappa], Q_\nu\rangle$*-generic set* q *so that* $q = \{\langle \tau, q_\tau\rangle \mid \tau < \nu\}$, *and* q_τ *is an* ω*-closed unbounded subset of* A_τ *for every* $\tau < \nu$.

PROOF. It follows from Proposition 5.6 and Lemma 5.7. $\square$

For q as in Propostion 5.8, let us define $\varphi(q) = \{\langle j(\tau), q_\tau \cup \{\kappa\}\rangle \mid \tau < \nu\}$. Then $\varphi(q) \in Q_{j(\nu)} \cap N[\mathring{P}_{\kappa+1}]$ and it is a $\langle V[\mathring{P}_\kappa], Q_{j(\nu)} \restriction \kappa\rangle$-generic.

6. The precipitous ideal

This section is close to those of [7]. The proof of precipitousness is based on the ideas from [7]. If $\nu < \kappa^+$, then for $\mathring{P}_{j(\kappa)} \supseteq \mathring{P}_\kappa$ a V-generic subset of $P_{j(\kappa)}$, and for $q \in N[\mathring{P}_{\kappa+1}]$ which is a $\langle V[\mathring{P}_\kappa], Q_\nu \rangle$-generic, let us pick some $G_{j(\nu)}$, so that it is a generic subset of $Q_{j(\nu)}$ and $\varphi(q) \in G_{j(\nu)}$. Then the elementary embedding

$$j : V \to N$$

can be extended to elementary embeddings

$$j^* : V[\mathring{P}_\kappa] \to N[\mathring{P}_{j(\kappa)}]$$

and

$$j^{**} : V[\mathring{P}_\kappa, q] \to N[\mathring{P}_{j(\kappa)}, G_{j(\nu)}]$$

as follows:

$$j^*(K_{\mathring{P}_\kappa}(\underline{a})) = K_{\mathring{P}_{j(\kappa)}}(j(\underline{a})) \qquad \text{for } \underline{a} \text{ a } P_\kappa\text{-name.}$$

Also $j^{**}(K_{\langle \mathring{P}_\kappa, q\rangle}(\underline{a})) = K_{\langle P_{j(\kappa)}, G_{j(\nu)}\rangle}(j(\underline{a}))$ for $\underline{a}$ a $\langle P_\kappa, Q_\nu \rangle$-name.

For $\nu = \kappa^+$ we shall do as in [7]. Let us define a subordering Q^* of $j^*(Q_{\kappa^+})$ in $V[\mathring{P}_{j(\kappa)}]$. For $q \in j^*(Q_{\kappa^+})$ let $C_q = \{q' \in Q_{\kappa^+} \mid j^*(q') \leqq q\}$. Note that $j^* \restriction Q_{\kappa^+}$ agrees with the isomorphism φ from Lemma 5.7 since $j^* \restriction \kappa = \mathrm{id}$ and $j^* \restriction 0n = j \restriction 0n$. Now $Q^* = \{q \in j^*(Q_{\kappa^+}) \mid$ for some $\nu < \kappa^+$, $C_q \subseteq Q_\nu$ and C_q is a $\langle V[\mathring{P}_k], Q_\nu \rangle$-generic$\}$.

Then let C^* be a $\langle V[\mathring{P}_{j(\kappa)}], Q^* \rangle$-generic and $C = \{q \in Q_{\kappa^+} \mid j^*(q) \in C^*\}$. As in [7] C^* is a $\langle N[\mathring{P}_{j(\kappa)}], j^*(Q_{\kappa^+}) \rangle$-generic and C is a $\langle V[\mathring{P}_\kappa], Q_{\kappa^+} \rangle$-generic. Also j extends to $j^{**} : V[\mathring{P}_\kappa, C] \to N[\mathring{P}_{j(\kappa)}, C^*]$.

Following [7], let us define I_ν for $\nu < \kappa^+$, as follows: For $x \in V[\mathring{P}_\kappa, C \restriction \nu]$, $x \in I_\nu$ iff there are $p \in \mathring{P}_\kappa$ and $q \in C \restriction \nu$,

$$p \Vdash_{P_{j(k)}} (\text{for every } \langle V[\mathring{P}_\kappa], Q_{j(\nu)} \restriction \kappa \rangle\text{-generic } q' \text{ with } q' \geqq j(\underline{q}),\ q' \Vdash_{Q_{j(\nu)}} \check{\kappa} \notin j(\underline{x})),$$

where $\underline{x}$, $\underline{q}$ are names of x and q.

Let $I = \bigcup_{\nu<\kappa^+} I_\nu$.

LEMMA 6.1. *I is the ideal of ω-nonstationary subsets of $\aleph_2$ (i.e., the sets whose complement are ω-closed unbounded subsets of $\aleph_2$).*

PROOF. First let us show that every ω-nonstationary set a belongs to I. Q_{κ^+} satisfies κ^+-c.c. so for some $\nu < \kappa^+$, $a \in V[\mathring{P}_\kappa, C \restriction \nu]$ and there is $b \in V[\mathring{P}_\kappa, C \restriction \kappa]$ s.t. $b \cap a = \emptyset$ and b is an ω-club in $V[\mathring{P}_\kappa, C \restriction \nu]$. Notice that since

Q_{κ^+} does not change cofinalities b remains ω-closed unbounded in $V[\breve{P}_\kappa, C]$. Let $\langle p, q\rangle \in \breve{P}_\kappa * C$ force that b contains an ω-club. Then, since

$$\emptyset \Vdash_{P_{j(\kappa)} * j(Q_{\kappa^+})} (\mathrm{cf}\, \check{\kappa} = \aleph_0 \quad \text{and}\ \underset{\sim}{b} = j(\underset{\sim}{b}) \cap \check{\kappa}),$$

$$p \Vdash_{P_{j(\kappa)}} (\text{for every } q' \text{ a } \langle V[\breve{P}_\kappa], Q_{j(\nu)} \,|\, \kappa\rangle\text{-generic with } q' \geqq j(\underset{\sim}{q}) q' \Vdash_{Q_{j(\nu)}} \check{\kappa} \in j(\underset{\sim}{b})).$$

Now let us show the converse, i.e., for every $a \in I$, $b = \kappa - a$ contains an ω-club. Suppose that $a \in I_\nu$ for some $\nu < \kappa^+$, i.e., there is a $\langle p, q\rangle \in P^*_\kappa * C_\nu$ such that $p \Vdash_{j(P_\kappa)}$ (for every q' a $\langle V[\mathring{P}_\kappa], Q_{j(\nu)} \restriction \kappa\rangle$-generic with $q' \geqq j(\underset{\sim}{q})' \Vdash_{Q_{j(\nu)}} \check{\kappa} \in j(\underset{\sim}{b})$). It is a statement in N. So, if

$$R = \{\alpha < \kappa \,|\, p \Vdash_{P_\kappa} (\text{for every } q' \text{ a } \langle V[P_\alpha], Q_\nu \restriction \alpha\rangle\text{-generic with } q' \geqq \underset{\sim}{q}\, q' \Vdash_{Q_\nu} \check{\alpha} \in \underset{\sim}{b})\},$$

then R belongs to $\mathcal{U}$. Note that, since P_κ satisfies κ-c.c., for some α large enough $q \in Q_\nu \restriction \alpha$. Let us assume that every element of R is bigger than this α.

Now let us consider for every $\gamma < \kappa^+$ the γ-th coordinate of C, i.e., let

$$t_\gamma = \bigcup \{q_\gamma \,|\, \text{for some } q \in C\, \langle \gamma, q_\gamma\rangle \in q\}.$$

Let $t = \{\langle \gamma, t_\gamma\rangle \,|\, \gamma < \nu + \omega + 1\}$. Then $t_{\nu+\omega}$ is an ω-closed unbounded subset of $A_{\nu+\omega}$. By (2) of the main definition for every $\alpha \in t_{\nu+\omega}$,

$$t \restriction \langle \nu + \omega, \alpha\rangle = \{\langle \tau, t_\tau \cap \alpha\rangle \,|\, \tau \in i''_{\nu+\omega}(\alpha)\}$$

is a $\langle V[\breve{P}_\alpha], Q_{\nu+\omega} \restriction \alpha\rangle$-generic. Lemma 4.4 implies $t \restriction \langle \nu, \alpha\rangle$ is a $\langle V[\breve{P}_\alpha], Q_\nu \,|\, \alpha\rangle$-generic, since $\nu \in i''_{\nu+\omega}(\alpha) = i''_\nu(\alpha) \cup \{\nu + n \,|\, n \in \omega\}$. Now for every $\alpha \in t_{\nu+\omega} \cap R$, $t \restriction \langle \nu, \alpha\rangle$ is stronger than our q. Hence $t \restriction \langle \nu, \alpha\rangle \Vdash_{Q_\nu} \check{\alpha} \in \underset{\sim}{b}$. But $t \restriction \langle \nu, \alpha\rangle \in C$. Hence $\alpha \in b$. So $b \supseteq t_{\nu+\omega} \cap R$. $R \in \mathcal{U}$, hence one of its subsets R_1 appears in the enumeration $\langle A_\nu \,|\, \nu < \kappa^+\rangle$ on some stage δ, i.e. $R_1 = A_\delta$. But then t_δ is an ω-closed unbounded subset of R. So $t_{\nu+\omega} \cap t_\delta$ is an ω-closed unbounded subset of b. □

LEMMA 6.2. *I is a precipitous ideal on $\aleph_2$ in $V[\mathring{P}_\kappa, C]$.*

See [7] for the proof.

Part II. The Closed Unbounded Filter Over $\aleph_2$

In this part we prove the following:

THEOREM II. *"ZFC + there is a normal measure concentrating on measurable* *cardinals" is consistent iff "ZFC + the closed unbounded filter on* $\aleph_2$ *is precipitous".*

Let us first prove the implication from left to right. We are starting from a model of ZFC + GCH with a measurable cardinal κ and two normal ultrafilters $\mathcal{U}_0$, $\mathcal{U}_1$ on it, so that $\mathcal{U}_0$ belongs to the ultrapower $N_1 \simeq V^\kappa/\mathcal{U}_1$. Let $B \in \mathcal{U}_1 - \mathcal{U}_0$ be a subset of $\{\alpha < \kappa \mid \alpha \text{ is measurable}\}$. Pick for every $\alpha \in B$ a normal ultrafilter $\mathcal{U}_\alpha$ so that the function $f(\alpha) = \mathcal{U}_\alpha$ represents $\mathcal{U}_0$ in N_1. Then $X \in \mathcal{U}_0$ iff $\{\alpha \in B \mid X \cap \alpha \in \mathcal{U}_\alpha\} \in \mathcal{U}_1$. W.l.o.g. suppose that for every $\alpha \in B$, $B \cap \alpha \notin \mathcal{U}_\alpha$.

Let us explain the idea. For X in $\mathcal{U}_0$ and Y in $\mathcal{U}_1$, we would like to shoot a club through $X \cup Y$. If we do it straight, then cardinals are collapsed. So we shall do some preparation. It goes as in Theorem I, only instead of the diamond we use the sequence of ultrafilters $\langle \mathcal{U}_\alpha \mid \alpha \in B\rangle$. After this is done, we can shoot clubs without collapsing any cardinals. The kind of iteration that we shall use is as in Theorem I. The ultrafilter $\mathcal{U}_i$ will be used to show that the ideal $\mathrm{NS}_{\aleph_2} \cap \{\alpha < \aleph_2 \mid \operatorname{cf} \alpha = \aleph_i\}$ is precipitous for $i = 0, 1$.

1. The preparation forcing

As in part I we define a revised countable support iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i \mid i < \kappa\rangle$, $|P_i| \leqq \aleph_{i+1}$. If i is not a strongly inaccessible cardinal, then $\underset{\sim}{Q}_i$ is the Levy collapse of $2^{\aleph_1}$ to $\aleph_1$ by countable conditions. If i is a strongly inaccessible and i does not belong to B (B is defined above), then $\underset{\sim}{Q}_i = \mathrm{Nm}'_{\aleph_2,\aleph_3}$. For $i \in B$, $\underset{\sim}{Q}_i = P^*\{\mathcal{U}_i\}$, where $P^*\{\mathcal{U}_i\}$ will be the set of all pairs $\langle c, A\rangle$ so that (1) c is an ω-closed subset of i, (2) for every limit point β of c, $c \cap \beta$ intersects with every closed unbounded subset of β, which belongs to $V[\mathring{P}_\beta]$; (3) $A \in \mathcal{U}_i$.

The ordering on $P^*\{\mathcal{U}_i\}$ is defined as follows: $\langle c_1, A_1\rangle \geqq \langle c_2, A_2\rangle$ if c_1 is an end-extension of c_2, $A_1 \subseteq A_2$ and $c_1 - c_2 \subseteq A_2$.

Let $P_\kappa = R \lim \bar{Q}$.

The next lemma is the analog of Lemma I.2.2. See [5] for the proof.

Lemma 1.1. *$P^*\{\mathcal{U}\}$ satisfies the strong $\mathbb{I}$-condition for a set $\mathbb{I}$ of monotone families so that $\mathcal{U}_i \in \mathbb{I}$.*

Lemma 1.2. *If $i \in B$, then every function $f \in V[\mathring{P}_{i+1}]$ from ω into $V[P_i]$ belongs to $V[P_i]$.*

Proof. The proof is as that of Lemma 2.3, only we shall consider elementary submodels $\mathcal{M}_\beta$ of $\langle V_{i^{+++}}, \in, P_i, \mathcal{U}_i, \underset{\sim}{f}\rangle$ s.t. $|\mathcal{M}_\beta| < i$. Then on a club C,

$\mathcal{M}_\beta \cap i = \beta$. Let $A_\beta = \bigcap\{A \in \mathcal{M}_\beta \mid A \in \mathcal{U}_i\}$, then $A_\beta \in \mathcal{U}_i$ and $\Delta A_\beta = \tilde{A} \in \mathcal{U}$. Let us pick $\alpha \in \tilde{A} \cap C$, α a limit point of C. Then for every $A \in \mathcal{M}_\alpha \cap \mathcal{U}_i$ for some $\beta < \alpha$, $\beta \in C$, $A \supseteq A_\beta$ since $\mathcal{M}_\alpha = \bigcup_{\beta<\alpha}\mathcal{M}_\beta$. Hence $\alpha \in A$. Now we continue as in Lemma 2.3. □

Also note that $P^*\{\mathcal{U}_i\}$ satisfies i^+-c.c. since $\langle c, A_1\rangle$ and $\langle c, A_2\rangle$ are always compatible.

2. The main forcing

Following I.4 let us fix enumerations by nonlimit ordinals $\langle A_{\nu+1} \mid \nu < \kappa^+\rangle$ of $\{A \in \mathcal{U}_0 \mid A \subseteq \kappa - B$ and every $\alpha \in A$ is an inaccessible$\}$ and $\langle B_{\nu+1} \mid \nu < \kappa^+\rangle$ of $\{B' \in \mathcal{U}_1 \mid B' \subseteq B\}$. Let us now define A_ν and B_ν for a limit $\nu < \kappa^+$. First we shall do it for $\nu < \kappa$. Let $\bar{A}_\nu = \bigcap_{\mu<\nu} A_\mu$ and $\bar{B}_\nu = \bigcap_{\mu<\nu} B_\mu$. Put $\bar{B}_\nu^{(1)} = \{\beta \in \bar{B}_\nu \mid \bar{A}_\nu \cap \beta \in \mathcal{U}_\beta\}$ (it will be the analog of $(\bar{A}_\nu)_\diamond$) and $\bar{A}_\nu^{(1)} = \{\alpha \in \bar{A}_\nu \mid \bar{B}_\nu^{(1)} \cap \alpha$ is a stationary subset of $\alpha\}$. Let $B_\nu = \bar{B}_\nu^{(2)} =_{\mathrm{df}} \{\beta \in \bar{B}_\nu^{(1)} \mid \bar{A}_\nu^{(1)} \cap \beta \in \mathcal{U}_\beta\}$ and $A_\nu = \bar{A}_\nu^{(2)} =_{\mathrm{df}} \{\alpha \in \bar{A}_\nu^{(1)} \mid \bar{B}_\nu^{(2)} \cap \alpha$ is a stationary subset of $\alpha\}$.

Now for $\nu \geqq \kappa$, as in I.4 we define $\bar{A}_\nu$ and $\bar{B}_\nu$, using $\langle \nu_\tau \mid \tau < \operatorname{cf}\nu\rangle$.

For $\langle \bar{A}_\nu, \bar{B}_\nu\rangle$ let us define, as above, $\langle \bar{A}_\nu^{(1)}, \bar{B}_\nu^{(1)}\rangle$ and $\langle \bar{A}_\nu^{(2)}, \bar{B}_\nu^{(2)}\rangle$. Put $A_\nu = \bar{A}_\nu^{(2)}$ and $B_\nu = \bar{B}_\nu^{(2)}$.

We shall use the notation of part I.

Main Definition II. For $\nu < \kappa^+$ we define in $V[\mathring{P}_\kappa]$ by induction the forcing notion Q_ν and the ordering $\leqq_\nu$ on it as follows:

An element $q \in Q_\nu$ is a sequence $\{\langle \alpha, q_\alpha\rangle \mid \alpha \in i''_\nu(\beta_q)\}$, where $\beta_q \in C_\nu$ so that

(1) for $\alpha \in i''_\nu(\beta_q)$, q_α is a closed subset of $A_\alpha \cup B_\alpha$;

(2) as in part I.

As in part I we define $\langle Q_{\kappa^+}, \leqq\rangle$. All the Lemmas 4.4–4.6 hold in our case.

The following analog of Proposition 5.1 holds,

Proposition 2.1. *For any limit ordinal $\nu < \kappa^+$ an ordinal $\alpha \in (A_\nu \cup B_\nu) \cap C_\nu$ and $p \in Q_\nu \restriction \alpha$, in the model $V[\mathring{P}_{\alpha+2}]$ there is a $\langle V[\mathring{P}_\alpha], Q_\nu \restriction \alpha\rangle$-generic set $q = \{\langle \tau, q_\tau\rangle \mid \tau \in i''_\nu(\alpha)\}$ so that $q \in Q_\nu$, $q_\nu \geqq p$ and $\bigcup(q_\tau \cap \alpha) = \alpha$ for every $\tau \in i''_\nu(\alpha)$.*

Proof. We prove this proposition by induction on $\langle \nu, \alpha\rangle$. Lemma 5.2 holds in our case. Only in Case 3 of this lemma shall we make a few changes. For $\alpha \in \bar{A}_\nu^{(1)}$, $\bar{B}_\nu^{(1)} \cap \alpha$ is stationary. So we can pick $\tau_n \in \bar{B}_\nu^{(1)} \cap (\mathrm{C}_n - \tau_{n-1}) \cap C_\nu$ s.t. $\tau_n \cap \mathrm{C}_n$ is unbounded in τ_n. The forcing on the stage τ_n is $P^*\{\mathcal{U}_{\tau_n}\}$.

Let $T=\{\langle c,A\rangle \mid c\subseteq \aleph_2, A\in \mathcal{U}_{\tau_n}\}$ be a generic subset of $P^*\{\mathcal{U}_{\tau_n}\}$. Let us define $T_1=\bigcup\{c\mid \exists A\langle c,A\rangle\in T\}$. Then T_1 is a closed unbounded in τ_n and from some place γ, $T_1-\gamma\subseteq \bar{A}_\nu$, since $\bar{A}_\nu\cap\tau_n\in\mathcal{U}_{\tau_n}$. Let $G_n=(T_1-\gamma)\cap(\mathbf{C}_n-\tau_n)\cap C_\nu$. The continuation is as in Lemma 5.2, only in the definition of t^n we shall add $\{\tau_n\}$ to every t^n_τ and check that such defined $t^n\in Q_\nu$. Note that since $\tau_n\in\bar{B}^{(1)}_\tau$, $\tau_n\in B_\mu$ for every $\mu\in i''_\mu(\tau_n)$. To show this, it is enough to prove the following:

Claim. *$t^n\restriction\langle\mu,\tau_n\rangle$ is $\langle V[\mathring{P}_{\tau_n}], Q_\mu\restriction\tau_n\rangle$-generic for every limit $\mu\in i''_\nu(\tau_n)$.*

For the proof note that G_n intersects every closed unbounded subset of τ_n in $V[\mathring{P}_{\tau_n}]$. So the arguments of part I work.

Now let us return to the proposition. If $\alpha\in A_\nu$, then in $V[\mathring{P}_{\alpha+1}]$, $\mathrm{cf}\,\alpha=\mathrm{cf}(\alpha^+)=\aleph_0$. We define $\langle \mathcal{M}^n_\beta\mid\beta<\alpha\rangle$, as in Proposition 5.1, but into $\mathcal{M}^n_0$ we include in addition $\langle B_\mu\mid\mu<\kappa^+\rangle$ and also for every $\gamma\in B_\mu$, $\mathcal{U}_\gamma$. We are picking a limit point γ_n of $E_n-\gamma_{n-1}$ which belongs to B_ν. It exists since $\alpha\in A_\nu=\bar{A}^{(2)}_\nu$ and so $B_\nu\cap\alpha$ is stationary in α. On step γ_n we forced with $P^*\{\mathcal{U}_{\gamma_n}\}$ and $\bar{A}^{(1)}_\nu\cap\gamma_n\in\mathcal{U}_{\gamma_n}$. Let $T\in V[\mathring{P}_{\gamma_n+1}]$ be its generic subset and $T_1=\bigcup\{c\mid\exists A\langle c,A\rangle\in\mathcal{U}_{\gamma_n}\}$. Then T_1 is a closed unbounded subset of γ_n and from some place γ, $\bar{A}^{(1)}_\nu\supseteq T_1-\gamma$. Let $G_n=E_n\cap(T_1-\gamma)$. We add $\{\gamma_n\}$ to q^n from Proposition 5.1, i.e. our q^n_τ is q^n_τ from Proposition 5.1 union with $\{\gamma_n\}$. Such $q^n\in Q_\nu$ and we continue as in Proposition 5.1.

Now suppose $\alpha\in B_\nu$. Then we force with $P^*\{\mathcal{U}_\alpha\}$. In $V[\breve{P}_{\alpha+1}]$, $|\alpha|=\aleph_1=\mathrm{cf}\,\alpha$ and $(\alpha^+)^V=\aleph_2$. If we do one more step then $(\alpha^+)^V$ also becomes of cardinality $\aleph_1$. So in $V[\breve{P}_{\alpha+2}]$ there is an enumeration $\langle D_i\mid i<\aleph_1\rangle$ of all dense subsets of $Q_\nu\restriction\alpha$ in $V[\mathring{P}_\alpha]$. Note that every countable subsequence of this sequence is in $V[\breve{P}_\alpha]$. It follows from Lemma 1.2. Let us define as above the elementary chain $\langle\mathcal{M}^i_\beta\mid\beta<\alpha\rangle$ of submodels of $\langle V_{\kappa^{+++}},\in,\alpha,\nu\rangle$, for every $i<\aleph_1$. But into $\mathcal{M}^i_0$ we include, instead of a name of $\underaccent{\tilde}{B}$, some P_α-name of D_i.

Let $E_i=\{\beta<\alpha\mid\mathcal{M}^i_\beta\cap\alpha=\beta$ and $\beta\in C_\nu\}$. Note that every $E_i\in V$, but $\{E_i\mid i<\aleph_1\}$ and probably some of its countable subsets does not. As in Lemma 5.3 for every inaccessible $\beta\in E_i$ $(i<\omega_1)$, $D_{i\beta}=_{\mathrm{df}}D_i\cap\mathcal{M}_\beta[\breve{P}_\beta]\in V[\breve{P}_\beta]$, it is a dense subset of $Q_\nu\restriction\beta$ and $Q_\nu\restriction\beta=Q_\nu\restriction\alpha\cap\mathcal{M}_\beta[\breve{P}_\beta]$.

Now $\alpha\in B_\nu$, hence $\bar{A}^{(1)}_\nu\cap\alpha\in\mathcal{U}_\alpha$. Let T be a generic subset of $P^*\{\mathcal{U}_\alpha\}$ and $T_1=\bigcup\{c\mid\exists A\langle c,A\rangle\in T\}$. From some place γ, $T_1-\gamma\subseteq\bar{A}^{(1)}_\nu$. Let $\{\alpha_i\mid i<\aleph_1\}$ be its increasing continuous enumeration.

Let us define in $V[\mathring{P}_{\alpha+1}]$ a sequence $\langle q_i\mid i<\aleph_1\rangle$, so that for every $i<\aleph_1$

(i) $q_i=\{\langle\tau,q_{i\tau}\rangle\mid\tau\in i''_\nu(\alpha_i)\}$,

(ii) $\max q_{i\tau}=\alpha_i$ for every $\tau\in i''_\nu(\alpha_i)$,

(iii) $q_i \in Q_\nu \cap V[\mathring{P}_{\alpha_i+1}]$,

(iv) $q_{i+1} \,{}_\nu\!\geqq q_i$,

(v) let δ_i be the first $\delta \leqq i$ so that for every $j < i$ $(\alpha_{j+1} \notin E_\delta)$ or $(\alpha_{j+1} \in E_\delta$ and $\delta_j \neq \delta)$.

If $\alpha_{i+1} \in E_{\delta_i}$, then q_{i+1} is stronger than some element of D_{δ_i}.

Note that to define $\langle q_j \mid j < i \rangle$ we need only $\langle \alpha_j \mid j < i \rangle$ and $\langle E_j \cap \alpha_i \mid j < i \rangle$. Both sequences belong to $V[\mathring{P}_{\alpha_i+1}]$, since

$$|\alpha_i|^{V[\mathring{P}_{\alpha_i+1}]} = |(\alpha_i^+)^{V[P_{\alpha_i}]}|^{V[P_{\alpha_i+1}]} = \aleph_1$$

and $P_\kappa / \mathring{P}_{\alpha_i+1}$ does not add reals.

So $\langle q_j \mid j < i \rangle \in V[\mathring{P}_{\alpha_i+1}]$. If i is a limit ordinal, then let $q_i = \{\langle \tau, q_{i\tau} \rangle \mid \tau \in i''_\nu(\alpha_i)\}$ where $q_{i\tau} = \bigcup \{q_{j\tau} \mid j < i \text{ and } \tau \in i''_\nu(\alpha_j)\} \cup \{\alpha_i\}$. Since $\{\alpha_j \mid j < i\}$ intersects every closed unbounded subset of α_i in $V[\mathring{P}_{\alpha_i}]$, $q_i \in Q_\nu$ (see the claim in Case 3, Lemma 5.2). If $i = j + 1$, then if $\alpha_{i+1} \notin E_{\delta_i}$ let q_{i+1} be any element satisfying (i)–(iv), otherwise $\alpha_{i+1} \in E_{\delta_i}$ and so $D_{\delta_i} \cap \mathcal{M}_{\alpha_{i+1}}[\mathring{P}_{\alpha_{i+1}}] = D_{\delta_i \alpha_{i+1}} \in V[\breve{P}_{\alpha_{i+1}}]$ and it is a dense subset of $Q_\nu \restriction \alpha_{i+1}$. Let us pick some $p \geqq_\nu q_i$ from this set and by the analog of Lemma 5.2 find $q_{i+1} \geqq_\nu p$ which satisfies (i)–(iii).

Let now $q = \{q_\tau \mid \tau \in i''_\nu(\alpha)\}$, where $q_\tau = \bigcup \{q_{i\tau} \mid \tau \in i''_\nu(\alpha_i)\} \cup \{\alpha\}$, for $\tau \in i''_\nu(\alpha)$. It remains to show that such defined q belongs to $Q_\nu \cap V[\mathring{P}_{\alpha+2}]$ and it is a $\langle V[\mathring{P}_\alpha], Q_\nu \restriction \alpha \rangle$-generic. The first half holds since we defined q inside $V[\mathring{P}_{\alpha+2}]$ and since $\{\alpha_i \mid i < \aleph_1\}$ intersects every closed unbounded subset of α in $V[\mathring{P}_\alpha]$. Let us prove the second half. So let D be a dense subset of $Q_\nu \restriction \alpha$ in $V[P_\alpha]$. Then D is some D_δ from the list of such subsets. It is enough to show that for some $i < \aleph_1$, $\delta_i = \delta$ and $\alpha_{i+1} \in E_\delta$. But it must hold since $E_\delta \in V$ and it is closed unbounded in α. Hence from some place j_δ every α_i with $i \geqq j_\delta$ belongs to E_δ. The same is true for every $\xi < \delta$. So in $V[\mathring{P}_{\alpha+2}]$ we have the countable sequence $\langle j_\xi \mid \xi \leqq \delta \rangle$. Since the cofinality of α is $\aleph_1$, there is $j < \aleph_1$ so that $\alpha_j \geqq \alpha_{j_\xi}$ for every $\xi \leqq \delta$. Now using (v) enough times, we obtain that for some $i \geqq j$, $\delta_i = \delta$ and since $\alpha_{i+1} \in E_\delta$, q_{i+1} will be stronger than some element of D_δ. □

As in part I the following holds:

Proposition 2.2. *For every limit $\nu < \kappa^+$ and an ordinal $\alpha \in (A_\nu \cup B_\nu) \cap C_\nu$ the forcing $Q_\nu \restriction \alpha$ over $V[\mathring{P}_\alpha]$ does not add new functions on $\aleph_1$ into $V[\mathring{P}_\alpha]$.*

Let $N_i \simeq V^\kappa / \mathcal{U}_i$ and $j_i : V \to N_i$ be the elementary embedding, for $i = 0, 1$.

Note that $\mathring{P}_{\kappa+1}$ has different meanings in N_0 and N_1 since in N_0, $P_{\kappa+1}$ is $P_\kappa * \mathrm{Nm}'_{\kappa,\kappa^+}$ but in N_1, $P_{\kappa+1}$ is $P_\kappa * P^*\{\mathcal{U}_0\}$.

Proposition 2.3. *For every limit $\nu < \kappa^+$ and $i = 0, 1$ in $N_i[\mathring{P}_{\kappa+1+i}]$ there is a*

$\langle V[\mathring{P}_\kappa], Q_{j_i(\nu)} \restriction \kappa\rangle$-generic set q_i so that $q_i \in Q_{j_i(\nu)}$, $q_i = \{\langle \tau, q_{i\tau}\rangle \mid \tau \in j_i''(\nu)\}$ and $\bigcup (q_{i\tau} \cap \kappa) = \kappa$ for every $\tau \in j_i''(\nu)$.

See part I for the proof. Note only that $\kappa \in j_0(A_\nu \cap C_\nu)$ and $\kappa \in j_1(B_\nu \cap C_\nu)$.

Lemma 2.4. *For every limit $\nu < \kappa^+$, $i = 0, 1$ and $\alpha \in (A_\nu \cup B_\nu) \cap C_\nu$ or $\alpha = \kappa$, $Q_\nu \restriction \alpha$ is isomorphic to $Q_{j_i(\nu)} \restriction \alpha$ and this isomorphism is in N_i.*

See Lemma 5.6. We define φ_i $(i = 0, 1)$ as in this lemma.

Proposition 2.5. *For every limit $\nu < \kappa^+$ and $i = 0, 1$ in $N_i[\mathring{P}_{\kappa+1+i}]$ there is a $\langle V[\mathring{P}_\kappa], Q_\nu\rangle$-generic set q_i so that $q_i = \{\langle \tau, q_{i\tau}\rangle \mid \tau < \nu\}$, $q_{i\tau}$ is a closed unbounded subset of $A_\tau \cup B_\tau$, all its points of cofinality $\aleph_0$ are in A_τ and of cofinality $\aleph_1$ in B_τ.*

The proof follows from Proposition 2.3 and Lemma 2.4.

For q_i as in the proposition let us define $\varphi_i(q_i) = \{\langle j_i(\tau), q_{i\tau} \cup \{\kappa\}\rangle \mid \tau < \nu\}$. Then $\varphi_i(q_i) \in Q_{j_i(\nu)} \cap N_i[\mathring{P}_{\kappa+1+i}]$ and it is a $\langle V[\mathring{P}_\kappa], Q_{j_i(\nu)} \restriction \kappa\rangle$-generic.

3. $\mathrm{NS}_{\aleph_2}$ is a precipitous ideal

Let us denote by $\mathrm{NS}_{\aleph_2}^{\aleph_i}$ the ideal of $\aleph_i$-nonstationary subsets of $\aleph_2$ (i.e. the sets whose complement contains an $\aleph_i$-closed unbounded set), where $i = 0, 1$.

A set $x \subseteq \aleph_2$ is $\aleph_i$-stationary if it intersects every $\aleph_i$-closed set. A set $x \subseteq \aleph_2$ is stationary iff for some $i \in 2$, $x \cap \{\alpha < \aleph_2 \mid \mathrm{cf}\, \alpha = \aleph_i\}$ is $\aleph_i$-stationary. So $\mathrm{NS}_{\aleph_2}$ is precipitous iff both $\mathrm{NS}_{\aleph_2}^{\aleph_0}$ and $\mathrm{NS}_{\aleph_2}^{\aleph_1}$ are precipitous.

Let us prove that $\varnothing \Vdash_{P_\kappa * Q_{\kappa^+}} (\mathrm{NS}_{\aleph_2}$ is precipitous). Otherwise, some $\langle p, q\rangle \in P_\kappa * Q_{\kappa^+}$ for some $i = \{0, 1\}$ force that $\mathrm{NS}_{\aleph_2}^{\aleph_i}$ is not precipitous.

Let us show that it is impossible if $i = 1$; the case $i = 0$ is the same.

As in [7] we pick a generic subset $\mathring{P}_\kappa * C$ of $P_\kappa * Q_{\kappa^+}$ and $\mathring{P}_{j_1(\kappa)} * C^*$ of $P_{j_1(\kappa)} * Q_{j_1(\kappa^+)}$, so that $\langle p, q\rangle \in \mathring{P}_\kappa * C$. The elementary embedding j_1 extends to $j_1^{**} : V[\mathring{P}_\kappa, C] \to N[\mathring{P}_{j_1(\kappa)}, C^*]$.

We define in $V[\mathring{P}_\kappa, C]$ ideals I_ν for $\nu < \kappa^+$, as follows:

For $x \in V[\mathring{P}_\kappa, C \restriction \nu]$, $x \in I_\nu$ iff there are $t \in \mathring{P}_\kappa$ and $r \in C \restriction \nu$, $t \Vdash_{P_{j_1(\kappa)}}$ (for every $\langle V[\mathring{P}_\kappa], Q_{j_1(\nu)} \restriction \kappa\rangle$-generic q' with $q' \geqq j_1(\underset{\sim}{r})$, $q' \Vdash_{Q_{j_1(\nu)}} \check{\kappa} \notin j_1(\underset{\sim}{x})$), where $\underset{\sim}{x}$, $\underset{\sim}{r}$ are names of x and r.

Let $I = \bigcup_{\nu < \kappa^+} I_\nu$.

Lemma 3.1. *I is the ideal of $\aleph_1$-nonstationary subsets of $\aleph_2$.*

See part I, Lemma 6.1 for the proof. Note that $t_{\nu+\omega}$ (from this lemma) will be a closed unbounded subset of $\aleph_2$ and since $R \in \mathcal{U}_1$, on some stage δ we shoot a club t_δ through $R \cap B \cup A_\delta$. Then $B \cap R \cap t_\delta$ will be an $\aleph_1$-club, since every ordinal in A_δ is of cofinality $\aleph_0$ in $V[\mathring{P}_\kappa, C]$.

Now by [7] I is a precipitous ideal on $\aleph_2$ in $V[\mathring{P}_\kappa, C]$. But we proved that $I = \mathrm{NS}^{\aleph_1}_{\aleph_2}$. Contradiction. So $\varnothing \Vdash_{P_\kappa * Q_{\kappa^+}}$ ($\mathrm{NS}_{\aleph_2}$ is precipitous).

4. The strength of $\mathrm{NS}_{\aleph_2}$ is precipitous

We would like to show that if $\mathrm{NS}_{\aleph_2}$ is precipitous, then there is an inner model with a measurable cardinal of order 2 (i.e., measurable with a normal measure on measurable cardinals).

Let us prove a little more general statement:

Proposition 4.1. *If the ideal* $\mathrm{NS}^{\aleph_0}_{\aleph_2}$ *of* $\aleph_0$*-nonstationary sets is precipitous and there is also some normal precipitous ideal* I *on* $\aleph_2$ *s.t.* $\{\alpha < \aleph_2 \mid \operatorname{cf} \alpha = \aleph_0\} \in I$, *then there is an inner model with a measurable cardinal of order* 2.

Proof. Let us force with I-positive sets. Let G be a generic ultrafilter, $j : V \to M_G$ the elementary embedding and M_G is the transitive collapse of $V \cap {}^{\aleph_2}V/G$.

Lemma 4.2. (i) *For every* $\alpha < \aleph_2$, $j(\alpha) = \alpha$.

(ii) $j(\aleph_2) > \aleph_2$.

(iii) $[\mathrm{id}]_G = \aleph_2$, *where* $\mathrm{id}(\alpha) = \alpha$ *for* $\alpha < \kappa$.

(iv) $\operatorname{cf}^{M_G}(\aleph_2^V) = \aleph_1$.

(v) *For every* $A \subseteq \aleph_2$, $A \in V$ *implies* $A \in M_G$.

(vi) *If* A *is an* ω*-closed subset of* $\aleph_2$ *in* V *then* A *is such also in* M_G.

Proof. See [6] for (i)–(iii). (iv) holds since $\{\alpha < \aleph_2 \mid \operatorname{cf} \alpha = \aleph_1\} \in G$. For (v) note that the function $\alpha \to A \cap \alpha$ represents A in M_G. (vi) holds since A is an ω-closed subset of $\aleph_2$ iff for every $\alpha < \aleph_2$, if $A \cap \alpha$ is unbounded in α and $\operatorname{cf} \alpha = \aleph_0$, then $\alpha \in A$. Also for $\alpha < \aleph_2$, $\operatorname{cf}^V \alpha = \operatorname{cf}^{M_G} \alpha$. ☐ of the lemma.

Suppose that there is no inner model with a measurable of order 2. We shall use Mitchell's Core Model for sequences of measures, see [11].

Our assumption implies that there is a sequence $\mathcal{F}$, so that any elementary embedding $i : K(\mathcal{F}) \to M$, with M a transitive class, is an iterated ultrapower of the core model $K(\mathcal{F})$. Then $i_G \restriction K(\mathcal{F}) : K(\mathcal{F}) \to K(\mathcal{F}')$ is an iterated ultrapower of $K(\mathcal{F})$ and $K(\mathcal{F}')$ is the core model for M_G. By our assumption $\aleph_2^V$ cannot be measurable in $K(\mathcal{F}')$. Let C be the filter of ω-closed unbounded subsets of $\aleph_2^V$ in M_G.

Claim. $C \cap K(\mathcal{F}') = ($*the filter of* ω*-closed unbounded subsets of* $\aleph_2$ *in* $V) \cap K(\mathcal{F}) = \mathcal{F}(\aleph_2^V, 0)$.

PROOF. First note that (the filter of ω-clubs in $V) \cap K(\mathscr{F}) = \mathscr{F}(\aleph_2^V, 0)$. Since $\mathrm{NS}_{\aleph_2}^{\aleph_0}$ is precipitous and if we force with its positive sets, then we obtain a nontrivial elementary embedding $j : K(\mathscr{F}) \to N$ with critical point $\aleph_2^V$. So it must be an iterated ultrapower of $K(\mathscr{F})$ using some ultrafilter on $\aleph_2^V$. But we assumed that there is only one such ultrafilter $\mathscr{F}(\aleph_2^V, 0)$. So every one of its elements belongs to every generic subset of $\mathrm{NS}_{\aleph_2}^{\aleph_0}$. Hence every $A \in \mathscr{F}(\aleph_2^V, 0)$ contains some ω-closed unbounded subset of $\aleph_2$ in V.

Now the statement of the claim follows from Lemma 4.2.

We are ready now to complete the proof. The filter C is a countably complete filter in M_G, hence the ultrapower

$$K(\mathscr{F}') \cap {}^{\aleph_2^V}K(\mathscr{F}')/\mathscr{F}(\aleph_2^V, 0)$$

is well founded. So in M_G we can define an elementary embedding $j : K(\mathscr{F}') \to M$ with a critical point $\aleph_2^V$. So $K(\mathscr{F}') \models \aleph_2^V$ is a measurable cardinal. Contradiction. □

Now it is natural to ask what happens if we replace the ideal $\mathrm{NS}_{\aleph_2}^{\aleph_0}$ by the ideal $\mathrm{NS}_{\aleph_2}^{\aleph_1}$ and the ideal I by the ideal s.t. $\{\alpha < \aleph_2 \mid \mathrm{cf}\,\alpha = \aleph_1\}$ belongs to it. Does this assumption imply a measurable of order 1? The answer is no.

PROPOSITION 4.3. *If there is a measurable cardinal, then there is a generic extension so that* $\mathrm{NS}_{\aleph_2}^{\aleph_1}$ *is precipitous and there is a normal precipitous* I *over* $\aleph_2$ s.t. $\{\alpha < \aleph_2 \mid \mathrm{cf}\,\alpha = \aleph_1\} \in I$.

Let us only describe the forcing notion and explain how it works.

We start with some measurable κ and two different normal ultrafilters $\mathscr{U}_0$ and $\mathscr{U}_1$ on it. It is possible to get such a model from the inner model of measurability, see [9]. Let A and B be some disjoint subsets of $\{\alpha < \kappa \mid \alpha \text{ is an inaccessible}\}$ so that $A \in \mathscr{U}_0$ and $B \in \mathscr{U}_1$. First we define a revised countable support iteration $\bar{Q} = \langle P_i, \underset{\sim}{Q}_i \mid i < \kappa \rangle$. If i is not in A, then $\underset{\sim}{Q}_i$ is the Levy collapse of $2^{\aleph_i}$ to $\aleph_1$ by countable conditions. If $i \in A$ then let $\underset{\sim}{Q}_i = \mathrm{Nm}'_{\aleph_2, \aleph_3}$ (see I for the definition). Let $P_\kappa = R \lim \bar{Q}$. Then P_κ does not add reals and for an inaccessible i, P_i satisfies i-c.c. In $V[\breve{P}_\kappa]$, $\mathscr{U}_i$ generates a pricipitous filter and it is concentrated on the set $\{\alpha < \aleph_2 \mid \mathrm{cf}\,\alpha = \aleph_i\}$, for $i = 0, 1$. Now as in [7] let us shoot ω_1-clubs through $\mathscr{U}_1$ and the filters generated by such shooting. In the last model $\mathscr{U}_0$ also can be extended to a precipitous filter. The point is that if $j_0 : V \to N_0 \simeq V^\kappa / \mathscr{U}_0$ and $j_0^* : V[\breve{P}_\kappa] \to N_0[\breve{P}_{j_0(\kappa)}]$, then in $N_0[\breve{P}_{j_0(\kappa)}]$ both κ and κ^+ are of cofinality ω. It gives the possibility (see I and II) to find a $V[\breve{P}_\kappa]$-generic subset of the forcing for shooting ω_1-clubs inside $N[\breve{P}_{\kappa+1}]$.

Added in proof. Recently M. Foreman, M. Magidor, S. Shelah and the author using different methods constructed models with NS_κ precipitous for $\kappa > \aleph_2$. On the other hand T. Jech [15] obtained results on the consistency strength of "NS_κ precipitous". But still the gap remains between the initial assumptions used in the models with NS_κ precipitous and the bounds of [15].

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